Variations on the factorization method for inverse scattering from penetrable media

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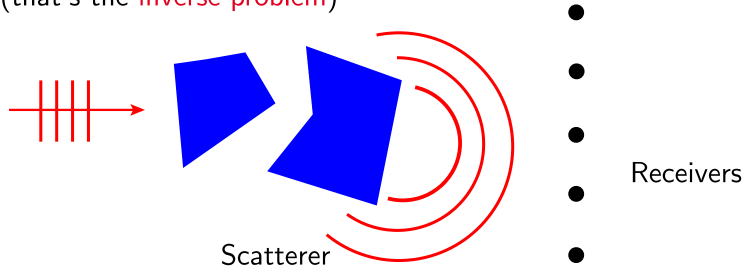
Overview

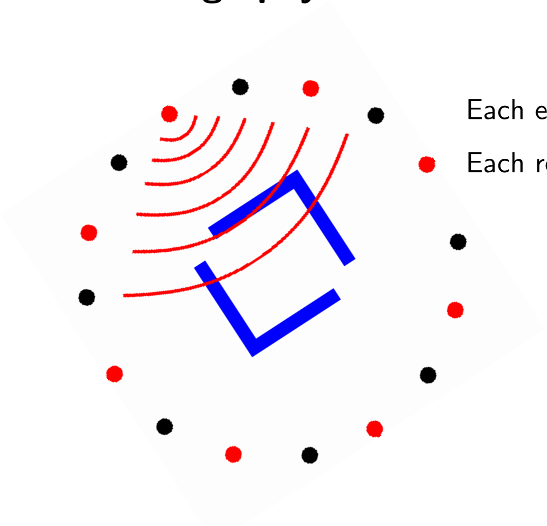
Inverse Problems for Waves

What is it all about?

- Send waves to an object with unknown properties
- Measure scattered waves
- Task: Deduce information about the object (that's the inverse problem)



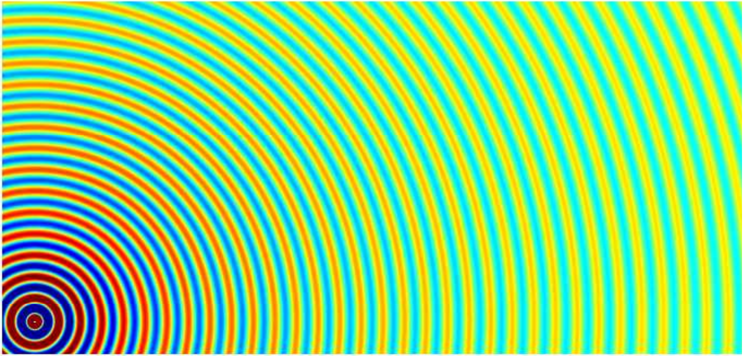
Acoustic tomography



Each emitter sends out waves

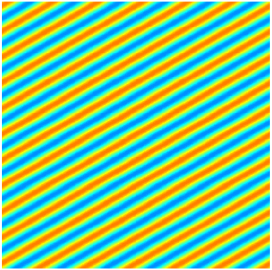
Each receiver records signals

Incident waves

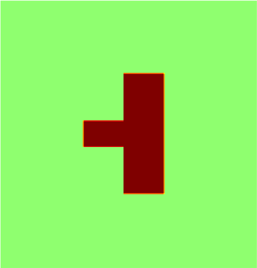


Time dependence $\exp(-\mathrm{i}\omega t)$, $k = \omega / c_0 = 2\pi/$wavelength

Time-harmonic wave scattering

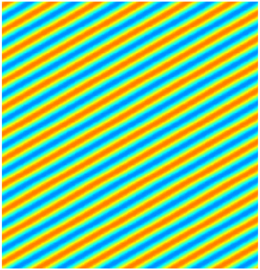


Incident field u^i



Scatterer D

Time-harmonic wave scattering



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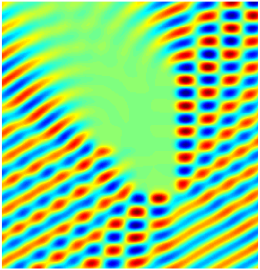
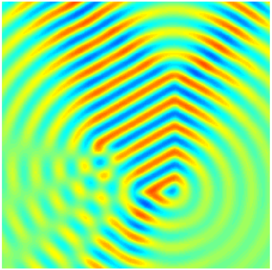
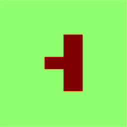


Total field u



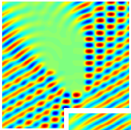
Scattered Field u^s

Wave scattering problems



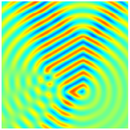
. . . are posed on exterior domains and link incident waves with outgoing scattered waves.

- Contrast $q > -1$, $\mathrm{supp}(q) = \overline{D}$ has no holes
- Incident field $u^i(x, \theta) = \exp(\mathrm{i}k\ x \cdot \theta)$, $\theta \in \mathbb{S}$, wave number k
- Total field $u(x, \theta) = u^i(x, \theta) + u^s(x, \theta)$ satisfies



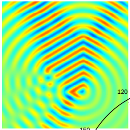
$$\Delta u + k^2(1 + q)u = 0 \text{ in } \mathbb{R}^d$$

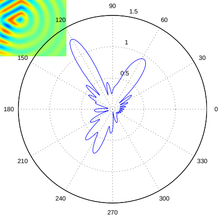
- Scattered field $u^s(\cdot, \theta) = u(\cdot, \theta) - u^i(\cdot, \theta)$ radiates:



$$|x|^{(d-1)/2} \left[\frac{\partial u^s}{\partial |x|}(x) - \mathrm{i}k u^s(x) \right] \to 0 \text{ as } |x| \to \infty$$

Far field pattern and far field operator



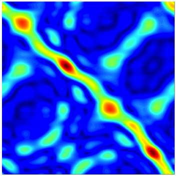


- Behavior of scattered field as $r \to \infty$:

$$u^s(r\hat{x},\theta) = \frac{e^{\mathrm{i}kr}}{4\pi r}\Big(u^\infty(\hat{x},\theta) + \mathcal{O}(r^{(1-d)/2})\Big)$$

- Far field pattern: $u^\infty(\hat{x},\theta)$ for $\hat{x}$, $\theta \in \mathbb{S}$
- Far field operator $F : L^2(\mathbb{S}) \to L^2(\mathbb{S})$,

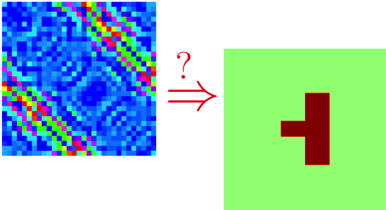
$$Fg = \int_{\mathbb{S}} u^\infty(\cdot,\theta) g(\theta)\,\mathrm{d}s$$



- Contrast q real-valued $\Rightarrow F \in \mathcal{K}(L^2(\mathbb{S}))$ is normal

Inverse scattering problem

- Direct scattering problem: Given q and θ, compute $u^\infty(\cdot, \theta)$! (linear, well-posed)
- Inverse scattering problem: Given $u^\infty(\hat{x}, \theta)$ for all $\hat{x}$, $\theta \in \mathbb{S}$, extract information on q! (non-linear, ill-posed)
- Finite data in practice: $N \times M$-matrix $(u^s(x_n, y_m))_{n,m=1}^{N,M}$
- Typical application: Full waveform inversion for seismic exploration of the earth
- More involved models: linear elasticity or electromagnetics



Uniqueness issues . . .

An obvious question is:

- Is q uniquely determined from far-field measurements?
- Answer: Yes . . .
- 3D: Nachman, Novikov, Ramm (all '88), $q \in L^\infty$
  Via unique continuation (Jerison & Kenig '85): $q \in L^p$, $p > 3/2$
- 2D: Bukhgeim '08, $q \in W^{1,p}$, $2 < p \leq \infty$,
  Blåsten '11: $q \in W^{\varepsilon,p}$, $\varepsilon > 0$
- Global assumption: $q \in L^\infty(\mathbb{R}^d)$ with compact support
- Numerical methods to extract information of far-field data?

Some inversion techniques

Given $u^\infty(\hat{x}, \theta)$ for $\hat{x}$, $\theta \in \mathbb{S}$, extract information on q!

- High/low frequency approximations (e.g. Born approximation)
  - $\Delta u^s(\cdot, \theta) + k^2(1+q)u^s(\cdot, \theta) = -k^2 q u^i(\cdot, \theta)$
  - Inverse problem becomes a linear, ill-posed problem.
- Newton-like schemes (Hohage '01)
- Contrast Source Inversion (Kleinman & van den Berg '92)
- Linear Sampling Method (Colton & Kirsch '96)
- Approximate Inverse (Abdullah & Louis '99)
  - Inverse scattering problem $\Rightarrow$ many inverse source problems
  - Precompute reconstruction kernels by SVD of forward operator
  - Determine q from nonlinear, algebraic equations

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Overview

Superposition principle

- Forward scattering problem linear & u^∞ depends linearly on u^s
- Superposition of incident plane waves (Herglotz wave):

$$u^i(x) = \int_{\mathbb{S}} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathsf{dS}(\theta)$$

- Scattered field equals

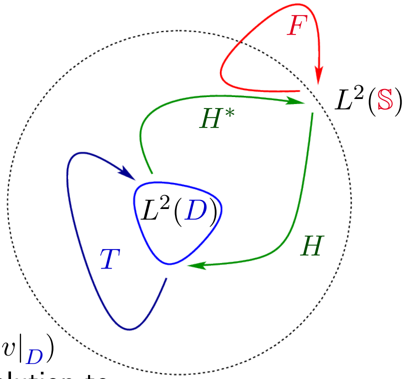
$$u^s(x) = \int_{\mathbb{S}} u^s(x,\theta) g(\theta)\, \mathsf{dS}(\theta)$$

- Far field equals

$$u^\infty(\hat{x}) = \int_{\mathbb{S}} u^\infty(\hat{x},\theta) g(\theta)\, \mathsf{dS}(\theta) = Fg$$

Factorization of F

- Superposition principle



- Herglotz operator $H : g \mapsto v_g|_D$

$$v_g(x) = \int_{\mathbb{S}} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathrm{dS}(\theta)$$

- Solution operator $T : f \mapsto q(f + v|_D)$
  where $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is radiating solution to

$$\Delta v + k^2(1+q)v = -k^2 q f \quad \text{in } \mathbb{R}^3$$

$F = H^* T H$

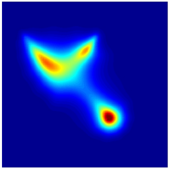
(Kirsch '98)

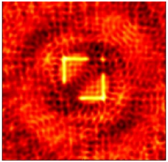
Note: Scattered field solves $\Delta u^s + k^2(1+q)u^s = -k^2 q u^i$

Factorization method

- ... are based on "self-adjoint" factorizations: $F = H^* T H$
- Note: $\Phi^\infty(\hat{x}, z) = e^{-\mathrm{i}k\,\hat{x}\cdot z}$ is far field of $\Phi(x,z) = \frac{e^{\mathrm{i}k|x-z|}}{4\pi|x-z|}$
- Two further ingredients:
  (1) Far field $\Phi^\infty(\cdot, z)$ belongs to range of H^* iff $z \in D$
  (2) If $\pm T$ is coercive + compact and injective
  $\Rightarrow$ range of H^* = range of $(F^* F)^{1/4}$
- Thus: $\Phi^\infty(\cdot, z) \in \mathrm{Rg}(F^* F)^{1/4}$ iff $z \in D$
- Picard's criterion:

$$z \in D \quad \Leftrightarrow \quad \left[\sum_{j \in \mathbb{N}} \frac{|\langle \Phi^\infty(\cdot, z), \psi_j \rangle|^2}{\mu_j} \right]^{-1} > 0$$

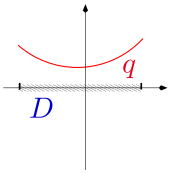


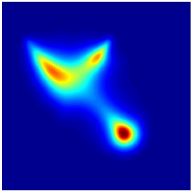


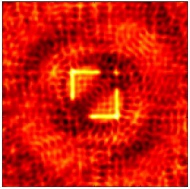
(Kirsch & Grinberg '08)

Factorization method

- $z \in D$ iff $\Phi^\infty(\cdot, z)$ belongs to range of $(F^* F)^{1/4}$
- Condition (2) for T: $\pm T$ is coercive + compact
- Sufficient requirement: Either $q > 0$ or $q < 0$ in D (no sign change!)
- Unclear how to extract information on values of q
- What about background media?







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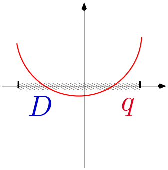

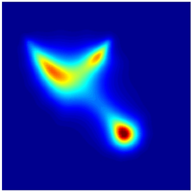

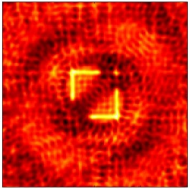

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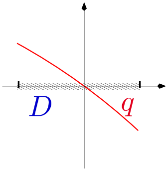

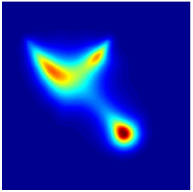

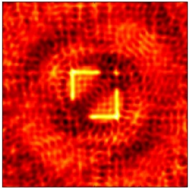

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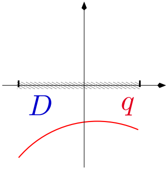
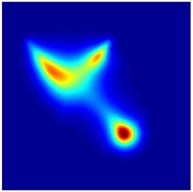
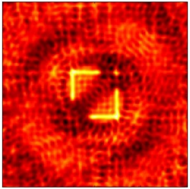






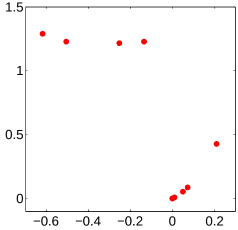
Take-home message

- "Monotonicity" between contrasts $q_{1,2}$ supported in $D \subset \mathbb{R}^d$ and spectrum of associated far field operators $F_{1,2}$. Roughly:

$$q_1|_{\partial D} \geq q_2|_{\partial D} \iff \text{eig'vals of } F_1 - F_2 \text{ tend to zero from right}$$

Consequences:

- Uniqueness of inverse problem for analytic q
- Monotonicity $\Rightarrow$ computable bounds for $q|_{\partial D}$
  Similar to EIT, Tamburrino & Rubinacci '02, Harrach & Ullrich '15



Behind the scene:

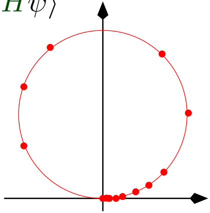
- Factorization of F via DtN operators

When tend eig'vals of F to zero from one side?

Theorem: If $F = H^* T H$ and $\pm \mathrm{Re}\, T$ is coercive + compact, then real parts of eig'vals μ_j of F tend to zero from the right/left.

- Assume: $\mathrm{Re}\, \langle +T\psi, \psi \rangle \geq c\|\psi\|_{\mathsf{X}}^2 - C\|\psi\|_{\mathsf{Y}}^2$ for $\mathsf{X} \overset{\text{cmp.}}{\hookrightarrow} \mathsf{Y}$.
  Denote linear hull of eig'funcs of F for eig'vals with negative real part by L^-. For $\psi \in L^-$,

$$0 \geq \mathrm{Re}\, \langle F\psi, \psi \rangle = \mathrm{Re}\, \langle H^* T H \psi, \psi \rangle = \mathrm{Re}\, \langle TH\psi, H\psi \rangle$$
$$\geq c\|H\psi\|_{\mathsf{X}}^2 - C\|H\psi\|_{\mathsf{Y}}^2$$



- Thus, $\|H\psi\|_{\mathsf{X}} \leq [C/c]^{1/2} \|H\psi\|_{\mathsf{Y}}^2$
- If $\dim L^- = \infty$ and H is injective:
  open mapping theorem $\Rightarrow$ contradiction

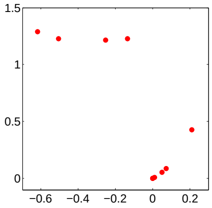
Scattering operator and far field difference

- Assumption: Contrasts $q_{1,2} \in L^\infty(\mathbb{R}^d)$ are real-valued
- Scattering operator $\mathcal{S}_{1,2} = I + 2\mathrm{i}|\gamma_d|^2 F_{1,2}$ is unitary on $L^2(\mathbb{S})$

$$\Rightarrow \mathcal{S}_2^*(F_1 - F_2) = \frac{1}{2\mathrm{i}|\gamma_d|^2}\mathcal{S}_2^*(\mathcal{S}_1 - \mathcal{S}_2) = \frac{1}{2\mathrm{i}|\gamma_d|^2}(\mathcal{S}_2^*\mathcal{S}_1 - I)$$

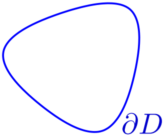
- As $(\mathcal{S}_2^*\mathcal{S}_1)^*\mathcal{S}_2^*\mathcal{S}_1 = I \Rightarrow \mathcal{S}_2^*(F_1 - F_2)$ is normal
- $\mathcal{S}_2^*(F_1 - F_2) = \sum_{j=1}^\infty \lambda_j \langle \cdot, g_j \rangle_{L^2(\mathbb{S})} g_j$

- 1st result by above factorization: If $q_1 - q_2 \gtrless 0$ in all of $\mathbb{R}^d \Rightarrow$ Eig'vals λ_j of $\mathcal{S}_2^*(F_1 - F_2)$ tend to zero from right/left: Re $\lambda_j \gtrless 0$ for large j



Factorization via Dirichlet-to-Neumann operators (I)

- Assumption: Contrast $q \in L^\infty(\mathbb{R}^d, \mathbb{R})$ with supp$(q) = \overline{D}$
- Herglotz operator $L : g \mapsto v_g|_{\partial D}$ for $v_g(x) = \int_{\mathbb{S}} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathrm{dS}$
- Exterior/interior DtN maps N^{out}, N_q^{in} between $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$



- $N^{\text{out}} : \psi \mapsto \partial v/\partial\nu$ where $v \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{D})$ is radiating weak solution to $\Delta v + k^2 v = 0$ in $\mathbb{R}^d \setminus \overline{D}$ with boundary values ψ
- $N_q^{\text{in}} : \psi \mapsto \partial v/\partial\nu$ where $v \in H^1(D)$ is weak solution to $\Delta v + k^2(1+q)v = 0$ in D with boundary values ψ

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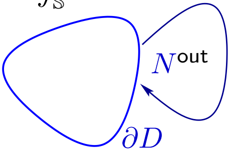

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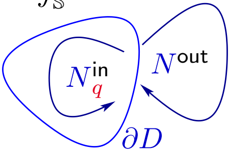

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Factorization via Dirichlet-to-Neumann operators (II)

- Consider incident Herglotz wave $u^i(x) = \int_{\mathbb{S}} e^{\mathrm{i}k\, x\cdot\theta} g(\theta)\, \mathsf{dS}(\theta)$
- $\Phi(x,y)$ radiating fundamental solution for Helmholtz equation
- Green's representation theorem in $\mathbb{R}^d \setminus \overline{D}$:

$$u^s = \int_{\partial D} - \Big[\quad \Big]\, \mathsf{dS}(y)$$

- Far field: $u^\infty = L^*\big(N_0^{\text{in}} - N^{\text{out}}\big)\big[u^s|_{\partial D}\big]$ in $L^2(\mathbb{S})$
- Finally: $u^s|_{\partial D} = (N_q^{\text{in}} - N^{\text{out}})^{-1}(N_0^{\text{in}} - N_q^{\text{in}})Lg$, such that

$$Fg = u^\infty = L^*\big(N_0^{\text{in}} - N^{\text{out}}\big)\big(N_q^{\text{in}} - N^{\text{out}}\big)^{-1}\big(N_0^{\text{in}} - N_q^{\text{in}}\big)L =: L^* M L$$

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Difference factorization and PDOs

- Assumption: Contrasts $q_{1,2} \in L^\infty(\mathbb{R}^d, \mathbb{R})$, supp$(q_{1,2}) = \overline{D}$
- $q_{1,2}$ are smooth in $\overline{D}$ and D is smooth domain
- Factorization $F_{1,2} = L^* M_{1,2} L$ implies

$$\mathcal{S}_2^*(F_1 - F_2) = L^*\big(M_1 - M_2 - 2\mathrm{i}k|\gamma_d|^2 M_2{}^* L L^* [M_1 - M_2]\big) L$$

- As $M_{1,2} = \big(N_0^{\text{in}} - N^{\text{out}}\big)\big(N_q^{\text{in}} - N^{\text{out}}\big)^{-1}\big(N_0^{\text{in}} - N_q^{\text{in}}\big)$, the principal symbol of $M_{1,2}$ on $\partial D \times T^*(\partial D)$ equals

$$(x, \xi^*) \mapsto \big(|\xi^*| - (-|\xi^*|)\big)\big(|\xi^*| - (-|\xi^*|)\big)^{-1}\big(k^2 q(x)/(2|\xi^*|)\big)$$

- Principal symbol of $M_1 - M_2$: $k^2(q_1(x) - q_2(x))/(2|\xi^*|)$

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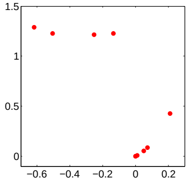
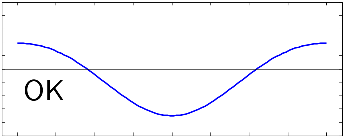
Concluding ...

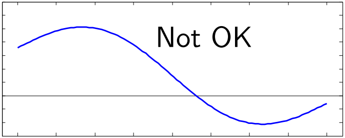


- Principal symbol of $M_1 - M_2$:
  $k^2(q_1(x) - q_2(x))/(2|\xi^*|)$
- If $q_1 \gtrless q_2$ on ∂D
  $\Rightarrow M_1 - M_2 = \text{sign}((q_1 - q_2)|_{\partial D})\, C + K$
  $\Rightarrow$ eig'vals of $\mathcal{S}_2^*(F_1 - F_2) = L^*[\text{sign}((q_1 - q_2)|_{\partial D})\, C + K]L$
  tend to zero from right/left
- If $q_1 - q_2$ takes both positive and negative values on ∂D
  $\Rightarrow \exists\, \infty$-many eigenvalues of $\mathcal{S}_2^*(F_1 - F_2)$ with positive and negative real part

Consequences

- If D is known, then F uniquely determines $q|_{\partial D}$
- If D is known and q is analytic, then F uniquely determines q in D
- Factorization method works for all contrasts q such that $q|_{\partial D}$ is sign-definite, independent of sign changes inside D





- Algorithmically: Exploit monotonicity for determining $q|_{\partial D}$ when D is known

Monotonicity algorithm when D is known

- Unknown contrast q, known far field operator F_q
- Test contrast q_{aux} with F_{aux}
- If eig'vals of $\mathcal{S}^*_{\text{aux}}(F - F_{\text{aux}})$ tend to zero from right (or left) $\Rightarrow q|_{\partial D} > q_{\text{aux}}|_{\partial D}$ (or $q|_{\partial D} > q_{\text{aux}}|_{\partial D}$)
- Easiest case: Test against constant $q_{\text{aux}} = c\,\mathbb{1}_D$
- Numerical criterion: $\sharp\{\lambda_j : \varepsilon \leq |\lambda_j| < 0.1, \operatorname{Re}\lambda_j \gtrless 0\}$

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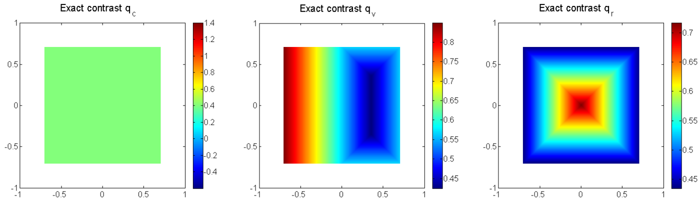

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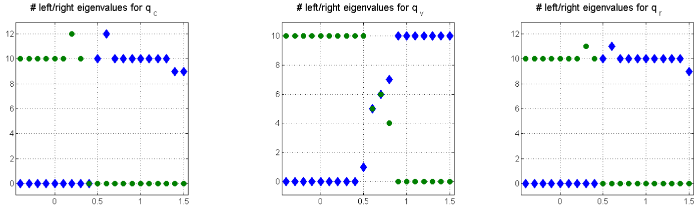

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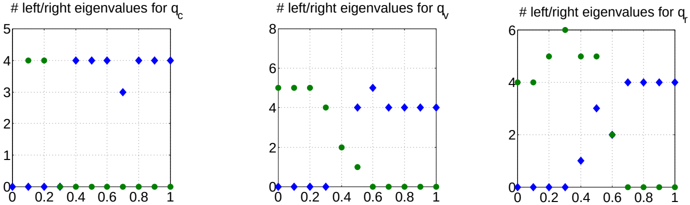

Monotonicity algorithm using linear test contrasts

- Unknown contrast q. Known far field operator F_q, domain D
- Linear test contrast q_{aux} with F_{aux}
- Parameterize via 16 pts on boundary, 11 slopes in normal direction, 15 offsets $\rightsquigarrow$ 2640 far fields ...
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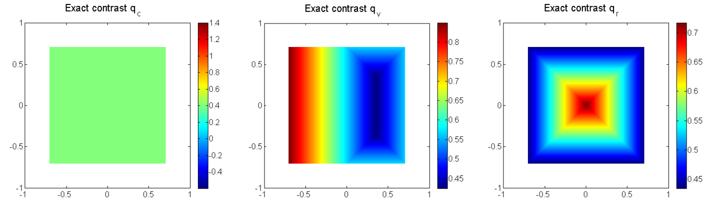

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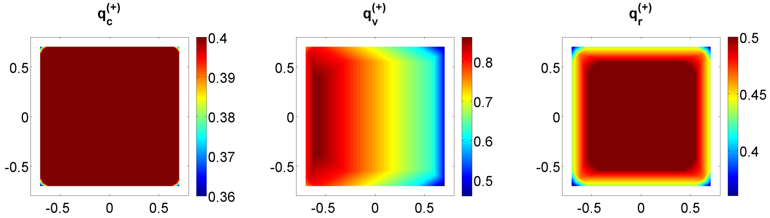

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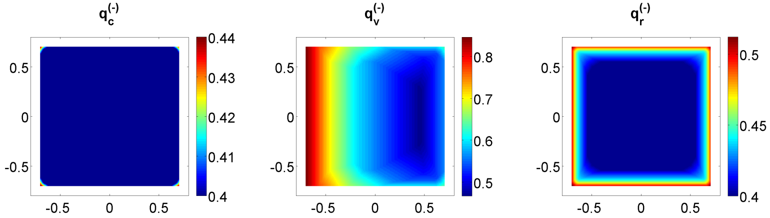
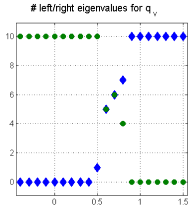


Summary & Extensions

- "Monotonicity" between contrasts $q_{1,2}$ supported in $D \subset \mathbb{R}^d$ and spectrum of associated far field operators $F_{1,2}$:

$$q_1|_{\partial D} \geq q_2|_{\partial D} \iff \text{eig'vals of } \mathcal{S}_2^*(F_1 - F_2) \to 0 \text{ from right}$$

- Factorization of F via DtN operators
- Computable bounds for $q|_{\partial D}$
- Avoid PDO-calculus by factorization as in Brühl '99
- References: Lakshtanov & L 2016
  Lakshtanov & Vainberg 2015;
  Cakoni & Harris 2015



Thanks for your attention!