

# Local ISS of Reaction-Diffusion Systems <sup>★</sup>

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**Abstract:** In the present paper we study local input-to-state stability of reaction-diffusion control systems by viewing them as control systems, based on differential equations in abstract spaces. We develop Lyapunov methods for verification of local input-to-state stability property and prove linearisation principle, that allows studying of local input-to-state stability of nonlinear systems from the stability properties of linearised systems. We illustrate theory on two examples of linear and nonlinear reaction-diffusion equations.

*Keywords:* Nonlinear control systems, distributed-parameter systems, stability analysis, linearization, Lyapunov methods.

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## 1. INTRODUCTION

The concept of input-to-state stability (ISS), widely used to study stability properties of control systems with respect to inputs, was introduced in Sontag (1989). Since that different methods were developed, that allow verifying of the input-to-state stability for finite-dimensional systems. A good survey about this topic is Sontag (2006). In particular, method of Lyapunov functions together with small-gain theorems (see Jiang et al. (1996), Dashkovskiy et al. (2007), Dashkovskiy et al. (2010), Karafyllis and Jiang (2009)) provides us with rich tools to investigate input-to-state stability of control systems.

Apart from systems, based on ordinary differential equations, the ISS concepts were applied to hybrid, switched and impulsive systems: Hespanha et al. (2008), Vu et al. (2007), Cai and Teel (2009). But for input-to-state stability of infinite-dimensional systems, with an important exception of time-delay systems (see, e.g. Pepe and Jiang (2006)), less attention was devoted. In Dashkovskiy and Mironchenko (2010) some basic results for certain classes of reaction-diffusion systems were presented.

In this paper we exploit semigroup theory methods and treat reaction-diffusion equations as differential equations in infinite-dimensional spaces. We investigate the local as well as global input-to-state stability property in abstract framework, and consider two examples of parabolic partial differential equations, linear and nonlinear, to show applicability of our methods.

The outline of the work is as follows: in Section 2 we introduce basic notions and notation. Then in Section 3 we discuss ISS for linear systems. Afterwards the method

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of ISS-Lyapunov functions is extended to the abstract control systems and the results are applied to the nonlinear reaction-diffusion equation. In Section 5 we prove the linearisation principle, that allows analysing of local input-to-state stability of nonlinear systems from the stability properties of linearised systems. In Section 6 we conclude the results of the paper and provide possible directions of future work.

## 2. PRELIMINARIES

Throughout the paper let  $(X, \|\cdot\|_X)$  and  $(U, \|\cdot\|_U)$  be state space and space of inputs, endowed with norms  $\|\cdot\|_X$  and  $\|\cdot\|_U$  respectively. For definiteness let them be Banach spaces.

For Banach spaces  $X, Y$  let  $L(X, Y)$  and  $L(X)$  be the spaces of bounded linear operators from  $X$  to  $Y$  and from  $X$  to  $X$  respectively. The norm in these spaces we denote by  $\|\cdot\|$ .

Let  $\mathbb{R}_+ := [0, \infty)$ .

We denote by  $U_c \subset \{f : \mathbb{R}_+ \rightarrow U\}$  the Banach space of admissible inputs and by  $U_c^{[s,t]} = \{f : [s,t] \rightarrow U : \exists g \in U_c, f(r) = g(r) \forall r \in [s,t]\}$  the restriction of  $U_c$  to the interval  $[s,t]$ .

Let  $\phi(t, s, x, u^{[s,t]}) \in X$  denote the state of a system at the moment  $t \in \mathbb{R}_+$ , if its state at the moment  $s \in \mathbb{R}_+$  was  $x \in X$  and on the time-span  $[s,t]$  the control  $u^{[s,t]} \in U_c^{[s,t]}$  was applied.

*Definition 1.* The triple  $\Sigma = (X, U_c, \phi)$  we call a control system, if the following holds:

- $\phi(t, t, x, \cdot) = x$  for all  $t \geq 0$ .
- $\forall t \geq r \geq s \geq 0, \forall x \in X, \forall u_1 \in U_c^{[s,r]}, u_2 \in U_c^{[r,t]}$  it holds  $\phi(t, r, \phi(r, s, x, u_1), u_2) = \phi(t, s, x, u)$ , where

$$u(\tau) := \begin{cases} u_1(\tau), \tau \in [s, r], \\ u_2(\tau), \tau \in [r, t]. \end{cases}$$

- for each  $x \in X$ ,  $u \in U_c$  the map  $t \rightarrow \phi(t, 0, x, u)$  is continuous
- $\phi$  is continuous w.r.t. two last arguments.

In this paper we consider time-invariant and forward-complete control systems. Time-invariance means, that the future evolution of a system depends only on the state of the system and of the applied input, but not on the initial moment of time. Forward completeness means, that  $\phi(t, 0, x, u)$  is defined for all  $x \in X, t \geq 0, u \in U_c^{[0, t]}$ .

*Definition 2.* For the formulation of stability properties the following classes of functions are useful:

$$\mathcal{K} := \{\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\}$$

$$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}$$

$$\mathcal{L} := \{\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t \rightarrow \infty} \gamma(t) = 0\}$$

$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K}, \forall t \geq 0, \beta(r, \cdot) \in \mathcal{L}, \forall r > 0\}$$

*Definition 3.* The control system  $(X, U_c, \phi)$  is *globally asymptotically stable at zero* uniformly with respect to  $x$  (0-UGAS $x$ ), if  $\exists \beta \in \mathcal{KL}$ , such that  $\forall \phi_0 \in X, \forall t \geq 0$  it holds

$$\|\phi(t, 0, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t). \quad (1)$$

If  $\beta$  can be chosen as  $\beta(r, t) = Me^{-at}r \forall r, t \in \mathbb{R}_+$ , for some  $a, M > 0$ , then  $(X, U_c, \phi)$  is called exponentially 0-UGAS $x$ .

Recall, that the 0-UGAS $x$  property is not equivalent in general to the 0-GAS (see Hahn (1967)), defined by:

1.  $\forall \varepsilon > 0 \exists \delta > 0 : \|x\|_X < \delta, t \geq 0 \Rightarrow \|\phi(t, 0, x, 0)\|_X < \varepsilon,$
2.  $\forall x \in X \|\phi(t, 0, x, 0)\|_X \rightarrow 0, t \rightarrow \infty.$

The norm in space  $U_c$  we denote as  $\|u\|_c$ .

To study the stability properties of control systems with respect to external inputs, we introduce the following notion

*Definition 4.* The control system  $(X, U_c, \phi)$  is called *locally input-to-state stable* (LISS), if  $\exists \rho_x, \rho_u > 0$  and  $\exists \beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$ , such that the inequality

$$\|\phi(t, t_0, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|u\|_c) \quad (2)$$

holds  $\forall \phi_0 : \|\phi_0\|_X \leq \rho_x, \forall t \geq 0$  and  $\forall u \in U_c : \|u\|_c \leq \rho_u$ .

If  $\beta$  can be chosen as  $\beta(r, t) = Me^{-at}r \forall r, t \in \mathbb{R}_+$ , for some  $a, M > 0$ , then  $(X, U_c, \phi)$  is called exponentially LISS (eLISS).

The control system is called *input-to-state stable* (ISS), if in the above definition  $\rho_x$  and  $\rho_u$  can be chosen equal to  $\infty$ . The notion of eISS is defined similarly.

Firstly we consider the case of linear systems.

### 3. LINEAR SYSTEMS

Let  $X$  be a Banach space,  $s(t) \in X$  and  $\mathcal{T} = \{T(t), t \geq 0\}$  be  $C_0$ -semigroup on  $X$  with an infinitesimal generator  $A = \lim_{t \rightarrow +0} \frac{1}{t}(T(t)x - x)$ .

Consider linear system with inputs:

$$\begin{aligned} \dot{s} &= As + f(u(t)), \\ s(0) &= s_0, \end{aligned} \quad (3)$$

where  $f : U \rightarrow X$  is locally Hölder continuous and space  $U_c$  is given by

$$U_c = \{g : \mathbb{R}_+ \rightarrow U : g \text{ is locally Hölder continuous}\}.$$

We assume also, that for some  $\gamma \in \mathcal{K}$  it holds

$$\|f(u)\|_X \leq \gamma(\|u\|_U), \forall u \in U.$$

The solution of (3) can be written in the form

$$s(t) = T(t)s_0 + \int_0^t T(t-r)f(u(r))dr. \quad (4)$$

For finite-dimensional systems it is well-known, that the system (3) is 0-GAS  $\Leftrightarrow$  (3) is e0-GAS  $\Leftrightarrow$  (3) is eISS  $\Leftrightarrow$  (3) is ISS. For infinite-dimensional systems we claim:

*Proposition 1.* The system (3) is exponentially 0-UGAS $x$  if and only if it is eISS.

**Proof.** The sufficiency follows from (2) for  $u \equiv 0$ .

System (3) is exponentially 0-UGAS $x \Leftrightarrow \mathcal{T}$  is an exponentially stable  $C_0$ -semigroup, that is,  $\exists M, w > 0$ , such that  $\|T(t)\| \leq Me^{-wt}$  for all  $t \geq 0$ .

From (4) we obtain

$$\begin{aligned} \|s(t)\|_X &\leq \|T(t)\| \|s_0\|_X + \int_0^t \|T(t-r)\| dr \gamma(\|u\|_c) \leq \\ &Me^{-wt} \|s_0\|_X + \frac{M}{w} \gamma(\|u\|_c), \end{aligned}$$

and the eISS property is verified. ■

*Remark 1.* One can prove, that for linear systems exponential 0-UGAS $x$  is equivalent to 0-UGAS $x$ .

Note, that in contrary to finite-dimensional systems, if a linear abstract control system is 0-GAS, then it does not imply, that the trajectories, corresponding to bounded (even arbitrary small) inputs are also bounded.

#### 3.1 Example: linear parabolic equations with Neumann boundary conditions

As an example consider a system of parabolic equations with Neumann conditions on the boundary.

Let  $G$  be a bounded domain in  $\mathbb{R}^p$  with smooth boundary, and let  $\Delta$  be Laplacian in  $G$ .

Let  $F : G \times U \rightarrow \mathbb{R}^n, F(x, 0) \equiv 0$ , and let  $u : G \times \mathbb{R}_+ \rightarrow U$  be the input. We assume, that  $F$  and  $u$  are continuous with respect to the first argument and Hölder continuous with respect to the second argument (uniformly with respect to the first one).

Now consider a parabolic system:

$$\begin{cases} \frac{\partial s(x, t)}{\partial t} - \Delta s = Rs + F(x, u(x, t)), & x \in G, t > 0, \\ s(x, 0) = \phi_0(x), & x \in G, \\ \frac{\partial s}{\partial n} \Big|_{\partial G \times \mathbb{R}_+} = 0. \end{cases} \quad (5)$$

Here  $\frac{\partial}{\partial n}$  is the normal derivative,  $s(x, t) \in \mathbb{R}^n, R \in \mathbb{R}^{n \times n}$ .

Let  $C(U)$  be the space of continuous and bounded (w.r.t. sup-norm) functions from  $U$  to  $U$ .

Let  $L : C(\bar{G}) \rightarrow C(\bar{G})$ ,  $L = -\Delta$  with

$$D(L) = \{f \in C^2(G) \cap C^1(\bar{G}) : Lf \in C(\bar{G}), \left. \frac{\partial f}{\partial n} \right|_{\partial G} = 0\}.$$

We define also the diagonal operator matrix  $A = \text{diag}(-L, \dots, -L)$  with  $-L$  as diagonal elements, with  $D(A) = (D(L))^n$ . The closure  $\bar{A}$  of  $A$  is an infinitesimal generator of an analytic semigroup on  $X = (C(\bar{G}))^n$ . Thus, the problem (5) may be considered as an abstract differential equation:

$$\begin{aligned} \dot{s} &= (\bar{A} + R)s + f(u(t)), \\ s(0) &= \phi_0, \end{aligned}$$

where  $f(u(t))(x) := F(x, u(x, t))$ .

One can check, that  $t \rightarrow f(u(t))$  is Hölder continuous, and inequality  $\|f(u)\|_X \leq \gamma(\|u\|_U)$  holds, with

$$\gamma(r) := \sup_{x \in G, u: \|u\|_U \leq r} F(x, u(x, t)).$$

Note, that  $\bar{A} + R$  also generates an analytic semigroup, as a sum of infinitesimal generator  $\bar{A}$  and bounded operator  $R$ .

Our claim is:

*Proposition 2.* System (5) is eISS  $\Leftrightarrow R$  is Hurwitz.

**Proof.** Denote by  $S(t)$  the semigroup, generated by  $\bar{A} + R$ .

We are going to find a simpler representation for  $S(t)$ . Consider (5) with  $u \equiv 0$ . Substituting in (5)  $s(x, t) = e^{Rt}v(x, t)$ , we obtain a simpler problem for  $v$ :

$$\begin{cases} \frac{\partial v(x, t)}{\partial t} = Av, & x \in G, t > 0, \\ v(x, 0) = \phi_0(x), & x \in G, \\ \left. \frac{\partial v}{\partial n} \right|_{\partial G \times \mathbb{R}_+} = 0. \end{cases} \quad (6)$$

In terms of semigroups, it means:  $S(t) = e^{Rt}T(t)$ , where  $T(t)$  is a semigroup, generated by  $\bar{A}$ . It is well-known (see, e.g. Henry (1981)), that the growth bound of  $T(t)$  is given by  $\sup \Re(\text{Spec}(\bar{A})) = \sup_{\lambda \in \text{Spec}(\bar{A})} \Re(\lambda)$ , where  $\Re(z)$  is real part of a complex number  $z$ .

We are going to find an upper bound of spectrum of  $\bar{A}$  in  $D(A)$ . Note, that  $\text{Spec}(A) = \text{Spec}(-L)$ . Thus, it is enough to estimate the spectrum of  $-L$ , that consists of all  $\lambda \in \mathbb{C}$ , such that the following equation has nontrivial solution

$$\begin{cases} Ls + \lambda s = 0, & x \in G \\ \left. \frac{\partial s}{\partial n} \right|_{\partial G} = 0. \end{cases} \quad (7)$$

Let  $\lambda > 0$  be an eigenvalue, and  $u_\lambda \neq 0$  be the corresponding eigenfunction. If  $u_\lambda$  attains its nonnegative maximum over  $\bar{G}$  in some  $x \in G$ , then, according to the strong maximum principle (see Evans (1998), p. 333),  $u_\lambda \equiv \text{const}$ , and consequently,  $u_\lambda \equiv 0$ ,  $\Rightarrow u_\lambda$  cannot be an eigenfunction. If  $u_\lambda$  attains the nonnegative maximum over  $\bar{G}$  in some  $x \in \partial G$ , then, by Hopf's lemma (see Evans (1998), p. 330),  $\frac{\partial u_\lambda(x)}{\partial n} > 0$ , also a contradiction. Consequently,  $u_\lambda \leq 0$  in  $\bar{G}$ . But  $-u_\lambda$  is also an eigenfunction, thus, applying the

same argument, we obtain, that  $u_\lambda \equiv 0$  in  $\bar{G}$ , thus  $\lambda > 0$  is not an eigenvalue.

Obviously  $\lambda = 0$  is an eigenvalue, therefore growth bound of  $T(t)$  is 0, and growth bound of  $S(t)$  is  $\omega_0 = \sup\{\Re(\lambda) : \exists x \neq 0 : Rx = \lambda x\}$ . So,  $R$  is Hurwitz is a sufficient condition for the system (5) to be exponentially 0-UGAS, and, consequently, eISS.

It is also necessary condition, because for constant  $\phi_0$  and  $u \equiv 0$  the solutions of (5) are for arbitrary  $x \in G$  the solutions of  $\dot{s} = Rs$ , and to guarantee the stability of these solutions  $R$  has to be Hurwitz. ■

#### 4. LYAPUNOV FUNCTIONS

To verify both local and global input-to-state stability of nonlinear systems, Lyapunov functions can be exploited. In this section we provide basic tools and illustrate them by the example.

*Definition 5.* A smooth function  $V : D \mapsto \mathbb{R}_+$ ,  $D \subset X$ ,  $0 \in \text{int}(D) = D \setminus \partial D$  is called local ISS-Lyapunov function (LISS-LF) for a control system  $(X, U, \phi)$ , if  $\exists \rho_x, \rho_u > 0$  and functions  $\psi_1, \psi_2 \in \mathcal{K}_\infty$ ,  $\chi \in \mathcal{K}$  and positive definite function  $\alpha$ , such that:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X$$

and  $\forall x \in X : \|x\|_X \leq \rho_x, \forall u \in U : \|u\|_U \leq \rho_u$  it holds:

$$\|x\|_X \geq \chi(\|u\|_U) \Rightarrow \dot{V}(x) \leq -\alpha(\|x\|_X), \quad (8)$$

where

$$\dot{V}(x) = \overline{\lim}_{t \rightarrow +0} \frac{1}{t} (V(\phi(t, 0, x, u)) - V(x)).$$

Function  $\chi$  is called ISS-Lyapunov gain for  $(X, U_c, \phi)$ .

If in the previous definition  $D = X$ ,  $\rho_x = \infty$  and  $\rho_u = \infty$ , then the function is called ISS-Lyapunov function.

*Theorem 1.* If a control system  $(X, U_c, \phi)$  possesses a LISS-Lyapunov function, then it is LISS.

The proof is similar to that from the original work Sontag (1989). For a counterpart of this theorem for infinite-dimensional dynamical systems (without controls) see, e.g., Henry (1981).

**Proof.** Let the control system  $\Sigma = (X, U_c, \phi)$  possesses a LISS-Lyapunov function and  $\psi_1, \psi_2, \chi, \alpha, \rho_x, \rho_u$  be as in the Definition 5. Take an arbitrary control  $u \in U_c$  with  $\|u\|_c \leq \rho_u$  such that

$$I = \{x \in D : \|x\|_X \leq \rho_x, V(x) \leq \psi_2 \circ \chi(\|u\|_c)\} \subset \text{int}(D).$$

Such  $u$  exists, because  $0 \in \text{int}(D)$ .

Firstly we prove, that  $I$  is invariant with respect to  $\Sigma$ , that is:  $\forall x \in I \Rightarrow x(t) = \phi(t, 0, x, u) \in I, t \geq 0$ .

If  $u \equiv 0$ , then  $I = \{0\}$ , and  $I$  is invariant, because  $x = 0$  is the equilibrium point of  $\Sigma$ . Consider  $u \neq 0$ .

If  $I$  is not invariant w.r.t.  $\Sigma$ , then, due to continuity of  $\phi$  with respect to  $t$ ,  $\exists t_0 > 0$ , such that  $V(x(t_0)) = \psi_2 \circ \chi(\|u\|_c)$ , and therefore  $\|x(t_0)\|_X \geq \chi(\|u\|_c)$ . Then from (8) it follows, that  $\dot{V}(x(t_0)) = -\alpha(\|x(t_0)\|_X) < 0$ . Thus, the trajectory cannot escape the set  $I$ .

Now take arbitrary  $x_0: \|x_0\|_X \leq \rho_x$ . As long as  $x_0 \notin I$ , we have the following differential inequality ( $x(t)$  is the trajectory, corresponding to the initial condition  $x_0$ ):

$$\dot{V}(x(t)) \leq -\alpha(\|x(t)\|_X) \leq -\alpha \circ \psi_2^{-1}(V(x(t))).$$

From the comparison principle (see Lin et al. (1996), Lemma 4.4 for  $y(t) = V(x(t))$ ) it follows, that  $\exists \tilde{\beta} \in \mathcal{KL}$ :  $V(x(t)) \leq \tilde{\beta}(V(x_0), t)$ , and consequently:

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t), \forall t : x(t) \notin I, \quad (9)$$

where  $\beta(r, t) = \psi_1^{-1} \circ \tilde{\beta}(\psi_2^{-1}(r), t)$ ,  $\forall r, t \geq 0$ .

From the properties of  $\mathcal{KL}$  functions it follows, that  $\exists t_1$ :

$$t_1 := \inf_{t \geq 0} \{x(t) = \phi(t, 0, x_0, u) \in I\}.$$

From the invariance of the set  $I$  we conclude, that

$$\|x(t)\|_X \leq \gamma(\|u\|_c), \quad t > t_1, \quad (10)$$

where  $\gamma = \psi_1^{-1} \circ \psi_2 \circ \chi \in K$ . Our estimates hold for arbitrary control  $u$ :  $\|u\|_c \leq \rho_u$ , thus, combining (9) and (10), we obtain the claim of the theorem. ■

*Remark 2.* Note, that if  $I \not\subset \text{int}(D)$ , then the trajectories can escape the set  $I$  through  $\partial D$ . In this way the condition  $I \subset D$  imposes the restrictions on possible values of  $u$ .

*Remark 3.* As a special case we have, that if a control system possesses an ISS-Lyapunov function, then it is ISS.

#### 4.1 Example

We are interested mainly in the study of abstract differential equations of the form

$$\dot{s} = As + f(s, u) \quad (11)$$

Here  $X$  is a Banach space,  $s(t) \in X$  and  $A : X \mapsto X$  is an infinitesimal generator of a  $C_0$ -semigroup,  $u \in U$  is an external input.

Let us consider the following example

$$\begin{cases} \frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - f(s) + u^m(x, t), & x \in (0, \pi), \quad t > 0, \\ s(0, t) = s(\pi, t) = 0. \end{cases} \quad (12)$$

We assume, that  $f$  is locally Lipschitz continuous, monotonically increasing up to infinity,  $f(-r) = -f(r)$  for all  $r \in \mathbb{R}$  (in particular,  $f(0) = 0$ ), and  $m \in (0, 1]$ .

For  $u$  it is enough to assume  $u^m(\cdot, t) \in L_2(0, \pi)$ , but if we are going to consider interconnections of system (12) with other systems, and use  $u$  as an input from other systems, then it makes sense to require from  $u$  some stronger properties (see Remark 4 below).

To reformulate (12) as an abstract differential equation we define  $As = \frac{d^2 s}{dx^2}$  with  $D(A) = H_0^1(0, \pi) \cap H^2(0, \pi)$ .

Here  $H^n(0, \pi)$  is a Sobolev space of  $f \in L_2(0, \pi)$ , which have weak derivatives of the order  $\leq n$  belonging to  $L_2(0, \pi)$  and  $H_0^1(0, \pi)$  is a closure of continuously differentiable functions with support, compact in  $(0, \pi)$  in the norm of  $H^1(0, \pi)$ . The norm on  $H_0^1(0, \pi)$  we define by

$$\|s\|_{H_0^1(0, \pi)} = \left( \int_0^\pi s_x^2(x) dx \right)^{\frac{1}{2}}.$$

Operator  $A$  generates an analytic semigroup on  $L_2(0, \pi)$ . System (12) takes form

$$\frac{\partial s}{\partial t} = As - f(s) + u^m, \quad t > 0. \quad (13)$$

Equation (13) defines the control system with state space  $X = H_0^1(0, \pi)$  and input space  $U = L_2(0, \pi)$ .

Consider the following ISS-Lyapunov function candidate:

$$V(s) = \int_0^\pi \left( \frac{1}{2} s_x^2(x) + \int_0^{s(x)} f(y) dy \right) dx. \quad (14)$$

We are going to prove, that  $V$  is an ISS-Lyapunov function.

Under made assumptions about function  $f$  it holds, that  $\int_0^\pi f(y) dy \geq 0$  for every  $r \in \mathbb{R}$ . Thus,  $V$  is positive definite:

$$V(s) \geq \int_0^\pi \frac{1}{2} s_x^2(x) dx = \frac{1}{2} \|s\|_{H_0^1(0, \pi)}^2 \quad (15)$$

Let us compute the Lie derivative of  $V$ :

$$\begin{aligned} \dot{V}(s) &= \int_0^\pi s_{xx}(x) s_{xt}(x) + f(s(x)) s_t dx = [s_x(x) s_t(x)]_{x=0}^{x=\pi} \\ &\quad + \int_0^\pi -s_{xx}(x) s_t(x) + f(s(x)) s_t(x) dx. \end{aligned}$$

From boundary conditions it follows  $s_t(0, t) = s_t(\pi, t) = 0$ . Thus, substituting expression for  $s_t$ , we obtain

$$\begin{aligned} \dot{V}(s) &= - \int_0^\pi (s_{xx}(x) - f(s(x)))^2 dx + \\ &\quad \int_0^\pi (s_{xx}(x) - f(s(x))) (-u^m) dx. \end{aligned}$$

Define

$$I(s) := \int_0^\pi (s_{xx}(x) - f(s(x)))^2 dx.$$

Using Cauchy-Schwarz inequality for the second term, we have:

$$\dot{V}(s) \leq -I(s) + \sqrt{I(s)} \|u^m\|_{L_2(0, \pi)}. \quad (16)$$

Now let us consider  $I(u)$

$$\begin{aligned} I(s) &= \int_0^\pi s_{xx}^2(x) dx - 2 \int_0^\pi s_{xx}(x) f(s(x)) dx + \\ &\quad + \int_0^\pi f^2(s(x)) dx = \int_0^\pi s_{xx}^2(x) dx + 2 \int_0^\pi s_x^2(x) \frac{\partial f}{\partial s}(s(x)) dx \\ &\quad + \int_0^\pi f^2(s(x)) dx \geq \int_0^\pi s_{xx}^2(x) dx. \end{aligned}$$

For  $s \in H_0^1(0, \pi) \cap H^2(0, \pi)$  it holds (see Henry (1981), p. 85), that

$$\int_0^\pi s_{xx}^2(x) dx \geq \int_0^\pi s_x^2(x) dx.$$

Overall, we have:

$$I(s) \geq \|s\|_{H_0^1(0, \pi)}^2. \quad (17)$$

Let us turn our attention to  $\|u^m\|_{L_2(0, \pi)}$ . Using Hölder inequality, we obtain:

$$\begin{aligned} \|u^m\|_{L_2(0, \pi)} &= \left( \int_0^\pi u^{2m} \cdot 1 dx \right)^{\frac{1}{2}} \leq \\ &\quad \left( \int_0^\pi u^2 dx \right)^{\frac{m}{2}} \left( \int_0^\pi 1^{1-\frac{1}{m}} dx \right)^{\frac{1-\frac{m}{2}}{2}} = \pi^{\frac{1-m}{2}} \|u\|_{L_2(0, \pi)}^m. \end{aligned} \quad (18)$$

Now we choose the gain as

$$\chi(r) = a\pi^{\frac{1-m}{2}} r^m, \quad a > 1.$$

If  $\chi(\|u\|_{L_2(0,\pi)}) \leq \|s\|_{H_0^1(0,\pi)}$ , we obtain from (16), using (18) and (17):

$$\begin{aligned} \dot{V}(s) &\leq -I(s) + \frac{1}{a} \sqrt{I(s)} \|s\|_{H_0^1(0,\pi)} \\ &\leq \left(\frac{1}{a} - 1\right)I(s) \leq \left(\frac{1}{a} - 1\right)\|s\|_{H_0^1(0,\pi)}^2. \end{aligned}$$

This inequality was verified for  $s \in D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$ . To prove the inequality for all  $s \in H_0^1(0,\pi)$  it is enough to mention, that

$$\frac{1}{t}(V(s(t)) - V(s_0)) = \dot{V}(s(t_*)) \leq \left(\frac{1}{a} - 1\right)\|s(t_*)\|_{H_0^1(0,\pi)}^2,$$

and  $s(t_*) \in D(A)$  (because  $A$  generates an analytic semigroup). Taking from the both parts the limit, when  $t \rightarrow +0$ , we obtain the needed estimation.

We have proved, that (14) is ISS-Lyapunov function, and consequently, (13) (with  $X = H_0^1(0,\pi)$ ,  $U = L_2(0,\pi)$ ) is ISS.

*Remark 4.* We have taken in the example  $U = L_2(0,\pi)$  and  $X = H_0^1(0,\pi)$ . But in case of interconnection with other parabolic systems (when we identify input  $u$  with the state of the other system), that have state space  $H_0^1(0,\pi)$  (as our system), we have to choose  $U = X = H_0^1(0,\pi)$ . In this case we can continue estimates (18), using Friedrichs' inequality

$$\int_0^\pi s^2(x)dx \leq \int_0^\pi s_x^2(x)dx$$

to obtain

$$\|u^m\|_{L_2(0,\pi)} \leq \pi^{\frac{1-m}{2}} \|u\|_{H_0^1(0,\pi)}^m \quad (19)$$

and choosing the same gains, prove the input-to state stability of (13) w.r.t. spaces  $U = X = H_0^1(0,\pi)$ .

## 5. LINEARISATION

In this section we prove the linearisation principle, that allows proving of the local ISS of a system, using solely information about ISS of the corresponding linearised system.

We assume, that  $X$  is a Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$ , and  $A$  generates an analytic semigroup on  $X$ .

Further, we suppose, that  $u : \mathbb{R}_+ \rightarrow U$  is Hölder-continuous, and  $f : X \times U \rightarrow X$ , defined on some open set  $Q$ ,  $(0,0) \in Q$  is Lipschitz continuous in first argument and Hölder continuous in second argument. That is, if  $(x,u) \in Q$ , then there exists a neighbourhood  $W$  of  $(x,u)$ :  $W \subset Q$ , such that  $\forall (y,v) \in W$  it holds

$$\|f(x,u) - f(y,v)\|_X \leq L(\|u - v\|_U^\theta + \|x - y\|_X),$$

for some constants  $L, \theta > 0$ .

Also we assume, that  $f(0,0) = 0$ , thus,  $x \equiv 0$  is an equilibrium point.

Consider a system

$$\dot{x} = Ax + f(x(t), u(t)), \quad x(t) \in X, u(t) \in U. \quad (20)$$

*Theorem 2.* Let in (20)

$$f(x,u) = Bx + Cu + g(x,u),$$

where  $B \in L(X)$  and  $C \in L(U, X)$  and for each  $w > 0 \exists \rho$ , such that  $\forall x : \|x\|_X \leq \rho, \forall u : \|u\|_U \leq \rho$  it holds

$$\|g(x,u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

If the system

$$\dot{x} = Ax + Bx + Cu \quad (21)$$

is eISS, then (20) is LISS.

**Proof.** Operator  $A$  is an infinitesimal generator of an analytic semigroup,  $B$  is bounded, therefore  $R = A + B$  is also a generator of an analytic semigroup.

System (21) is eISS, and consequently exponentially 0-UGAS, therefore there exists (see, e.g., Curtain and Zwart (1995)) the corresponding Lyapunov function for (21):

$$V(x) = \langle Px, x \rangle, \quad (22)$$

where  $P \in L(X)$  is a positive bounded operator, for which it holds, that

$$\langle Rx, Px \rangle + \langle Px, Rx \rangle = -\|x\|_X^2, \quad \forall x \in D(R). \quad (23)$$

We are going to prove, that  $V$  is LISS-Lyapunov function for the system (20) for properly chosen gains. Let us compute Lie derivative of  $V$  with respect to the system (20).

Firstly consider the case, when  $x \in D(R) = D(A)$ .

$$\begin{aligned} \dot{V}(x) &= \langle P\dot{x}, x \rangle + \langle Px, \dot{x} \rangle = \\ &\langle P(Rx + Cu + g(x,u)), x \rangle + \langle Px, Rx + Cu + g(x,u) \rangle = \\ &\langle P(Rx), x \rangle + \langle Px, Rx \rangle + \\ &\langle P(Cu + g(x,u)), x \rangle + \langle Px, Cu + g(x,u) \rangle. \end{aligned}$$

We continue estimates using the property

$$\langle P(Rx), x \rangle = \langle Rx, Px \rangle,$$

which holds for positive operators, inequality (23) and for the last two terms Cauchy-Schwarz inequality in space  $X$

$$\begin{aligned} \dot{V}(x) &\leq -\|x\|_X^2 + \|P(Cu + g(x,u))\|_X \|x\|_X + \\ &+ \|Px\|_X \|Cu + g(x,u)\|_X \leq -\|x\|_X^2 + \end{aligned}$$

$$\|P\| \| (Cu + g(x,u)) \|_X \|x\|_X + \|P\| \|x\|_X \|Cu + g(x,u)\|_X \leq -\|x\|_X^2 + 2\|P\| \|x\|_X (\|C\| \|u\|_U + \|g(x,u)\|_X)$$

For each  $w > 0 \exists \rho$ , such that  $\forall x : \|x\|_X \leq \rho, \forall u : \|u\|_U \leq \rho$  it holds

$$\|g(x,u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

Using this inequality, we continue estimates

$$\begin{aligned} \dot{V}(x) &\leq -\|x\|_X^2 + 2w\|P\| \|x\|_X^2 + 2\|P\| (\|C\| + w) \|x\|_X \|u\|_U \\ \text{Take } \chi(r) &:= \sqrt{r}. \text{ Then for } \|u\|_U \leq \chi^{-1}(\|x\|_X) = \|x\|_X^2 \\ \text{we have:} & \end{aligned}$$

$$\dot{V}(x) \leq -\|x\|_X^2 + 2w\|P\| \|x\|_X^2 + 2\|P\| (\|C\| + w) \|x\|_X^3. \quad (24)$$

Choosing  $w$  and  $\rho$  small enough, the right hand side can be estimated from above by some negative quadratic function of  $\|x\|_X$ .

These derivations hold for  $x \in D(R) \subset X$ . If  $x \notin D(R)$ , then for all admissible  $u$  the solution  $x(t) \in D(R)$  and  $t \rightarrow V(x(t))$  is a continuous differentiable function for all  $t > 0$  (these properties follow from the properties of solutions  $x(t)$ , see theorem 3.3.3 in Henry (1981)).

Therefore, by mean-value theorem,  $\forall t > 0 \exists t_* \in (0, t)$ :

$$\frac{1}{t}(V(x(t)) - V(x)) = \dot{V}(x(t_*)).$$

Taking the limit when  $t \rightarrow +0$ , we obtain, that (24) holds for all  $x \in X$ .

This proves, that  $V$  is LISS-Lyapunov function with  $\|x\|_X \leq \rho$ ,  $\|u\|_U \leq \rho$  and consequently, (20) is LISS. ■

In particular, the theorem holds for finite dimensional systems. In this case the assumptions about Hölder continuity for functions  $u$  and for  $f$  with respect to second argument (needed to achieve existence and uniqueness of solutions of (20) (see section 3.3 in Henry (1981) for  $\alpha = 0$ )) can be replaced with milder assumptions (see Alekseev et al. (1987), p. 183). We state it as

*Corollary 1.* Let a control system  $(\mathbb{R}^n, L_\infty(\mathbb{R}_+, \mathbb{R}^m), \phi)$  be given by the ODE system

$$\dot{x} = f(x, u), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \quad (25)$$

Let in (25)

$$f(x, u) = Bx + Cu + g(x, u),$$

where  $g(x, u)$  is as in Theorem 2.

If  $B$  is Hurwitz, then (25) is LISS.

Moreover, the Lyapunov function is given by  $V(x) = x^T Px$ , where  $P > 0$ ,  $PB + BP < 0$ . ■

Recall, that existence of such a Lyapunov function is equivalent to the fact, that  $B$  is Hurwitz (see, e.g., Sontag (1998)).

## 6. CONCLUSION

In the paper we have analysed local and global input-to-state stability of infinite-dimensional control systems. For these systems Lyapunov methods and linearisation principle have been developed. The results were illustrated on two examples from linear and semilinear reaction-diffusion equations. One of directions for future work is a generalisation of small-gain theorems for large-scale networks of finite-dimensional systems to infinite-dimensional case.

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