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Abstract

We consider the computation of Hermitian nonnegative definite solutions of algebraic Riccati equations. These solutions are the limit, $P = \lim_{i\to\infty} P_i$, of a sequence of matrices obtained by solving a sequence of Lyapunov equations. The procedure parallels the well-known Kleinman technique but the stabilizability condition on the underlying linear time-invariant system is removed. The convergence of the constructed sequence $\{P_i\}_{i\geq 1}$ is guaranteed by the minimality of P_i in the set of Hermitian nonnegative definite solutions of the Lyapunov equation in the ith iteration step.

Key words. algebraic Riccati equation, Newton's method, Kleinman iteration, Lyapunov equation, continuous-time linear systems.

1 Introduction

This paper is devoted to the analysis of algebraic Riccati equations (ARE) of the form

$$0 = A^*P + PA - PBB^*P + C^*C =: \mathcal{F}(P) \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and $P \in \mathbb{C}^{n\times n}$ is the Hermitian nonnegative definite solution to be determined.

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Equations of the form (1) appear in many areas of control theory; see, e.g., [BLW91, LR95, Meh91, Sim96]. The ARE is a quadratic matrix equation, i.e., the *Riccati function* $\mathcal{F}(X)$ is a quadratic, matrix-valued polynomial. The ARE (1) usually arises in continuous-time applications and is therefore also called the *continuous-time ARE* in order to distinguish it from the ARE related to control problems for discrete-time systems.

It is the purpose of this paper to provide an iterative method for the determination of the nonnegative definite solutions of the ARE without using the system-theoretic assumptions of stabilizability or detectability of the linear time-invariant system corresponding to the coefficient matrices A, B, C in (1). The proposed procedure can be seen as an extension of the well-known Kleinman technique [Kle68], where the nonnegative definite solution of (1) is obtained as the limit of nonnegative definite solutions of suitable Lyapunov equations. This iteration can be seen as applying Newton's method to the Riccati function $\mathcal{F}(X)$. Though there have been several extensions and generalizations of Newton's method for AREs in the literature (see, e.g., [AL84, BB98, GL98, LR95, Meh91]), all results for Newton's method so far assume the stabilizability of (A, B). We will show how this assumption can be removed at the cost of keeping the constant term in the ARE positive semidefinite such that it can always be expressed in the form C^*C as in (1). This iteration can thus be used to solve the linear-quadratic optimal control problem without stability as considered in [Gee88].

The following notation is adopted from [Wim94, Wim95] and will be used throughout this paper. The conjugate transpose matrix of a matrix M is denoted by M^* and $M \geq 0$ means that M is nonnegative definite. The abbreviation M^{-*} is used for $(M^*)^{-1}$. We use the usual ordering of Hermitian matrices, i.e., $M \geq N$ if and only if $M - N \geq 0$.

The complex plane is partitioned as

$$\mathbb{C} = \mathbb{C}_< \cup \mathbb{C}_= \cup \mathbb{C}_> \quad , \quad \mathbb{C}_\leq = \mathbb{C}_< \cup \mathbb{C}_= \quad , \quad \mathbb{C}_\geq = \mathbb{C}_= \cup \mathbb{C}_> ,$$

where the subscripts indicate the relation of the real parts to zero. For example, $\mathbb{C}_> = \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 0\}$.

The spectrum of a matrix $M \in \mathbb{C}^{n \times n}$ is denoted by $\sigma(M)$. For the generalized eigenspaces we use the notation

$$E_\lambda(M) = \sum_{i \geq 0} \ker (M - \lambda I)^i = \ker (M - \lambda I)^n$$

and the corresponding decompositions

$$\mathbb{C}^n = E_\geq(M) \oplus E_<(M) , \qquad E_\geq(M) = E_>(M) \oplus E_=(M) ,$$

where the subscripts refer to the real parts of the eigenvalues. For example, $E_\leq(M) = \bigoplus \{E_\lambda(M), \operatorname{Re}(\lambda) \leq 0\}$.

A matrix A is *stable* (or *Hurwitz*) iff $E_{\geq}(A) = \{0\}$ or in other words, if $\sigma(M)$ is contained in the open left half plane; (A, B) is *stabilizable* if there exists a matrix $F \in \mathbb{C}^{m \times n}$ such that $A - BF$ is stable; and (A, C) is *detectable* if there exists $L \in \mathbb{C}^{n \times p}$ such that $A - LC$ is stable. Moreover, from the modal characterization of stability and taking into account the invariance of the reachability and unobservability subspaces, the pair (A, B) is stabilizable if and only if

$$R(A, B)^{\perp} \cap E_{\geq}(A^*) = \{0\},$$

where $R(A, B)$ is the *reachability subspace* defined as

$$R(A, B) = \operatorname{Im}\left(B\,,\ AB\,,\ \ldots\,,A^{n-1}B\right)$$

and $\perp$ denotes the orthogonal complement. The matrix pair (A, B) is *controllable* if $\operatorname{rank}(R(A, B)) = n$. Analogously, the pair (A, C) is detectable if

$$V(A, C) \cap E_{\geq}(A) = \{0\},$$

where $V(A, C)$ is the *unobservability subspace*, given by

$$V(A, C) = \ker\left(C^T,\ (CA)^T\ \ldots\,,\ (CA^{n-1})^T\right)^T.$$

The matrix pair (A, C) is *observable* if $\operatorname{rank}(V(A, C)) = n$. Finally, it is convenient to define for each A-invariant subspace, $V \in \operatorname{Inv}(A)$, the sets

$$V_\lambda = V \cap E_\lambda(A)$$

and, analogous to the notation introduced above, $V_<$, $V_>$, $V_=$, and $V_\leq$, $V_\geq$.

We will frequently make use of the well-known *Kalman canonical form* of linear time-invariant systems [Kal62, Kal63, Kal82]. This form corresponds to the direct sum decomposition of $\mathbb{C}^n$ given by

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus V_3 \oplus V_4,$$

where $V_1 \oplus V_2 = R(A, B)$ and $V_1 \oplus V_3 = V(A, C)$. Using nonsingular state-space transformations only, it is therefore possible to transform (A, B, C) to the form

$$A = \begin{bmatrix} A_1 & A_{12} & A_{13} & A_{14} \\ 0 & A_2 & 0 & A_{24} \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ C_2^T \\ 0 \\ C_4^T \end{bmatrix}^T. \tag{2}$$

Originally, the Kleinman iteration arises from the computation of the stabilizing solution of (1), which, if it exists, is the only solution P yielding a stable closed-loop matrix $A - BB^*P$; see, e.g., [LR95]. In [Kle68], convergence of the method is proved for controllable and observable systems. Using that stabilizability and detectability of (A, B, C) are sufficient conditions for the existence of the stabilizing solution (see [Kuč72]), the convergence of the Kleinman iteration was also proved in this context in [San74]. Further generalizations yield the so far most general result given in [LR95] which is based only on stabilizability and the existence of a Hermitian solution to the inequality $\mathcal{F}(X) > 0$. There, Newton's method (or the Kleinman iteration) is used to prove the existence of the unique *maximal* (with respect to the ordering of Hermitian matrices) solution of the ARE where the constant term is allowed to be indefinite. It is shown that the iterates of the method converge to this solution. Note that the maximal and stabilizing solution coincide if the latter one exists.

The stabilizability of (A, B) and the stability of the initial matrix $A - BB^*P_0$ in the Kleinman process are two conditions which have been maintained as hypothesis in the literature. However, in some situations where (A, B) is not stabilizable the Kleinman iteration yields a non-increasing sequence of nonnegative definite matrices. Using the Kalman decomposition of the linear system defined by (A, B, C), we can prove a more general convergence result for the Kleinman iteration.

The outline of the paper is as follows. Section 2 contains auxiliary results and the well-known convergence theory for the Kleinman procedure in the context of stabilizable systems. Section 3 is devoted to extending this iterative linearization method to general, not necessarily stabilizable systems. In Section 4 the obtained results are used for giving new necessary and sufficient conditions for the existence of a particular nonnegative definite solution of the ARE (1). Some concluding remarks are given in Section 5.

2 The Kleinman Iteration for Stabilizable Systems

In this section we review the convergence theory of the Kleinman iteration for the case that (A, B) is stabilizable. First, we give some preliminary results. Lemmas 2.1–2.3 are used in Theorem 2.4 to obtain the convergence of the sequence of solutions of the Lyapunov equations to the stabilizing solution of the ARE.

The first lemma establishes a well-known result about the solutions of Lyapunov equations; see, e.g., [LT85].

Lemma 2.1 *If $A \in \mathbb{C}^{n\times n}$ is stable, then the Lyapunov equation*

$$A^*P + PA + C^*C = 0 \tag{3}$$

admits a unique solution which is nonnegative definite. This solution is *given by the formula*

$$P = \int_0^\infty \exp(A^*t)\ C^*C\ \exp(At)\ dt\ .$$

In the following, let $\mathcal{L}(A,C)$ denote the set of Hermitian solutions of (3) and $\mathcal{L}_\mathcal{N}(A,C)$ the subset of nonnegative definite matrices in $\mathcal{L}(A,C)$. Analogously, by $\mathcal{R}(A,B,C)$ and $\mathcal{R}_\mathcal{N}(A,B,C)$ we will denote the set of Hermitian and nonnegative definite Hermitian solutions of the ARE (1).

The following two lemmas can be found in [Won74].

Lemma 2.2 *If (A,C) is detectable, then for every $G \in \mathbb{C}^{m\times n}$ and every positive semidefinite $Q \in \mathbb{C}^{n\times n}$, the pair $(A+BG, C^*C+Q+G^*G)$ is also detectable.*

In particular, Lemma 2.2 holds for all (A,C) with A stable.

Lemma 2.3 *If (A,C) is detectable and $\mathcal{L}_\mathcal{N}(A,C) \neq \emptyset$, then A is stable.*

With the above results it is now possible to prove that the stabilizability of (A,B) is sufficient for the existence of the maximal solution in $\mathcal{R}_\mathcal{N}(A,B,C)$.

Theorem 2.4 (stabilizable systems) *If the system (A,B) is stabilizable, then choosing $F_0 \in \mathbb{C}^{m\times n}$ such that $A_0 = A - BF_0$ is stable, the sequence of solutions of the Lyapunov equations*

$$A_i^* P_{i+1} + P_{i+1}A_i + C_i^*C_i = 0, \tag{4}$$

where

$$\begin{aligned} F_i &= B^*P_i \qquad \textit{for } i \neq 0, & (5)\\ A_i &= A - BF_i, \quad \textit{and} & (6)\\ C_i^*C_i &= F_i^*F_i + C^*C \geq 0, & (7)\end{aligned}$$

satisfies the following assertions:

(i) *For each $i \geq 0$, the matrix A_i is stable and the Lyapunov equation (4) admits a unique solution which, moreover, is nonnegative definite, i.e.,*

$$\mathcal{L}(A_i,C_i) = \mathcal{L}_\mathcal{N}(A_i,C_i) = \{P_{i+1}\}\,.$$

(ii) *$\{P_i\}_{i\geq 1}$ is a non-increasing sequence satisfying $P_i \geq 0$ for all $i \geq 1$. Moreover, $\tilde{P} = \lim_{i\to\infty} P_i$ exists and $\tilde{P} \in \mathcal{R}_\mathcal{N}(A,B,C)$.*

(iii) *$\tilde{P} = \lim_{i\to\infty} P_i$ is maximal in $\mathcal{R}_\mathcal{N}(A,B,C)$ and $\mathcal{R}(A,B,C)$.*

(iv) If $\tilde{P}$ is stabilizing, i.e., $A - BB^*\tilde{P}$ is stable, then the convergence rate of the process is quadratic, that is

$$\|\tilde{P} - P_{i+1}\| \leq \gamma \cdot \|\tilde{P} - P_i\|^2 \quad \textit{for } i \geq 1,$$

where γ is a constant, independent of $\tilde{P}$.

Proof. A complete proof for the above result can be found, e.g., in [LR95]. We will sketch proofs of Parts (i)–(iii) as this is instructive for the theory derived in the next section.

Part *(i)* is proved by induction. Assuming the stability of A_i, which is true for $i = 0$, it is shown by Lemma 2.1 that $P_{i+1} \in \mathcal{L}_{\mathcal{N}}(A_i, C_i)$ exists. Moreover, $P_{i+1} \in \mathcal{L}_{\mathcal{N}}(A_{i+1}, \tilde{C}_{i+1})$ where

$$\tilde{C}_{i+1}^* \tilde{C}_{i+1} = (F_i - F_{i+1})^* (F_i - F_{i+1}) + C_{i+1}^* C_{i+1}. \tag{8}$$

Using Lemma 2.2 applied to the detectable matrix pair (A_i, C) with $G = (F_i - F_{i+1})$ and $Q = F_{i+1}^* F_{i+1}$, it follows that $(A_{i+1}, \tilde{C}_{i+1}^*, \tilde{C}_{i+1})$ and hence also $(A_{i+1}, \tilde{C}_{i+1})$ is detectable. Thus, Lemma 2.3 guarantees the stability of A_{i+1}.

Part *(ii)* is a consequence of the stability of A_i together with the fact that $D_i = P_i - P_{i+1} \in \mathcal{L}(A_i, F_{i-1} - F_i)$. Thus, $D_i \geq 0$ using Lemma 2.1. As $P_i \geq 0$ from Lemma 2.1, the P_i form a non-increasing sequence which is bounded from below. It follows that their limit, denoted by $\tilde{P}$, exists. Taking limits in (4) yields $\tilde{P} \in \mathcal{R}_{\mathcal{N}}(A, B, C)$.

For Part *(iii)* note that $P_{i+1} - P \in \mathcal{L}(A_i, F_i - B^*P)$ if $P \in \mathcal{R}(A, B, C)$. This implies, using Lemma 2.1, that $P_{i+1} - P \geq 0$ for all i and hence, $\tilde{P} = \lim_{i \to \infty} P_i$ is maximal in $\mathcal{R}(A, B, C)$. ■

As mentioned earlier, the Kleinman iteration is equivalent to Newton's method applied to the ARE (1). Re-arranging the standard formulae for Newton's method shows that P_{i+1} can be computed from P_i by solving the Lyapunov equation (4). Thus, the convergence theorems for the Kleinman iteration are only convergence theorems for a particular Newton algorithm and consequently the stabilizability of (A, B) is not necessarily required for the convergence of the process. We will give conditions under which the Kleinman iteration converges to a nonnegative definite solution of the ARE for possibly nonstabilizable systems.

3 The Kleinman Iteration for General Systems

In this section a sequence of nonnegative definite matrices $\{P_i\}_{i \geq 1}$ is defined using the same approach as in Theorem 2.4. However, the corresponding system is not necessarily stabilizable and consequently, the coefficient matrices A_i in the Lyapunov equations (4) are not necessarily stable. Thus, the Lyapunov equation may have no or infinitely many solutions.

Now, for the proof of the convergence in the general case, some additional results replacing Lemmas 2.1–2.3 and the existence of a minimal solution in $\mathcal{L}_{\mathcal{N}}(A, C)$ are also needed. These results as well as their proofs can for the most part be found in [Pas95] and will be proven here for the sake of completeness.

Lemma 3.1 $\mathcal{L}_{\mathcal{N}}(A, C) \neq \emptyset$ *if and only if the unstable part of* A *is unobservable, that is, if*

$$E_{\geq}(A) \subset \ker(C). \tag{9}$$

Moreover, there is a unique solution in $\mathcal{L}_{\mathcal{N}}(A, C)$ *if and only if*

$$E_{=}(A) = \{0\}. \tag{10}$$

Proof. The first part follows immediately from the fact that (1) has a positive semidefinite solution if and only if

$$V(A, C) + R(A, B) + E_{<}(A) = \mathbb{C}^n \tag{11}$$

(see [GH90] and also [Wim95]) by observing that (3) is a special case of (1) with $B = 0$.

Let $S \in \mathbb{C}^{n \times n}$ be nonsingular such that

$$\tilde{A} := S^{-1}AS = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \qquad \tilde{C} := CS = [\, C_1, \; C_2 \,], \tag{12}$$

where $\sigma(A_{11}) \subset \mathbb{C}_{\geq}$ and $\sigma(A_{22}) \subset \mathbb{C}_{<}$. Now $\mathcal{L}_{\mathcal{N}}(A, C) \neq \emptyset$ is equivalent to $\mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C}) \neq \emptyset$. Hence, we have $E_{\geq}(\tilde{A}) \subset \ker(\tilde{C})$ by the first part. This implies that $C_1 = 0$.

Pre-multiplying (3) by S^*, post-multiplying by S, and then partitioning $\tilde{P} := S^*PS = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]$ according to (12), the Lyapunov equation (3) now decouples into

$$\begin{aligned} A_{11}^*P_{11} + P_{11}A_{11} &= 0, & (13)\\ A_{11}^*P_{12} + P_{12}A_{22} &= 0, & (14)\\ A_{22}^*P_{21} + P_{21}A_{11} &= 0, & (15)\\ A_{22}^*P_{22} + P_{22}A_{22} &= -C_2^*C_2. & (16) \end{aligned}$$

As A_{22} is stable, (16) has a unique nonnegative solution by Lemma 2.1.

Now suppose that (10) holds. This implies $\sigma(A_{11}) \cap \mathbb{C}_{=} = \emptyset$ and by Sylvester's Theorem (see, e.g., [LT85, Theorem 12.3.2]), (13) has the unique nonnegative solution $P_{11} = 0$. So for any element in $\mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$, $P_{11} = 0$ and $P_{22} \geq 0$ are fixed. For a nonnegative definite matrix of this structure it follows necessarily that $P_{12} = 0$ and $P_{21} = 0$. Hence

$\mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C}) = \{S^{-*}\mathrm{diag}\,(0,\, P_{22})\, S^{-1}\}$, where P_{22} is the unique solution of (16).

On the other hand, assume there exists a unique positive semidefinite solution in $\mathcal{L}_{\mathcal{N}}(A, C)$ and $E_{=}(A) \neq \{0\}$. Then there exists $x \in \ker\,(A_{11} - \lambda I)$, $x \neq 0$, for some $\lambda \in \sigma\,(A_{11})$ with $\mathrm{Re}(\lambda) = 0$. Hence $\hat{P}_{11} := (\alpha x)(\alpha x^*)$ satisfies (13) for any $\alpha \in \mathbb{C}$. This implies that there are infinitely many solutions of (3) which contradicts the assumption. ■

To illustrate the above result, consider the following example.

Example 3.2 *Let* $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & a \end{smallmatrix}\right]$ *for* $a \geq 0$ *and* $C = [\,2,\ c\,]$ *for* $c \in \mathbb{R}$. *Then* $P_{11} = 1$ *and there can only be a nonnegative solution of the corresponding Lyapunov equation (3) if* $c = 0$. *If* $a \neq 0$, *this solution is unique and given by* $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *while for* $a = 0$, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right] \in \mathcal{L}_{\mathcal{N}}(A, C)$ *for all* $x \geq 0$.

In the following, the term *minimal solution* (or *maximal solution*) is used to denote the nonnegative definite solution which is minimal (or maximal) in $\mathcal{L}_{\mathcal{N}}\,(A, C)$. The same terminology is employed for solutions of the ARE in $\mathcal{R}_{\mathcal{N}}\,(A, B, C)$.

Lemma 3.3 *If* $\mathcal{L}_{\mathcal{N}}\,(A, C) \neq \emptyset$, *then a minimal solution in* $\mathcal{L}_{\mathcal{N}}\,(A, C)$ *exists and is unique.*

Proof. Without loss of generality we may assume that A is transformed via a similarity transformation such that $\tilde{A} := S^{-1}AS = \mathrm{diag}\,(A_{11}, A_{22}, A_{33})$, where $\sigma\,(A_{11}) \subset \mathbb{C}_{>}$, $\sigma\,(A_{22}) \subset \mathbb{C}_{=}$, and $\sigma\,(A_{33}) \subset \mathbb{C}_{<}$. Partitioning $\tilde{C} := CS = [\,C_1,\ C_2,\ C_3\,]$ accordingly, it follows from the first part of Lemma 3.1 that $C_1 = 0$ and $C_2 = 0$. Partitioning the solutions of (3) analogously as

$$\tilde{P} := S^*PS = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{21} & P_{31} \end{bmatrix}, \tag{17}$$

we can split the transformed equation (3) into nine equations. By inspecting these equations we can deduce from Sylvester's Theorem that $P_{ij} = 0$ for $(i, j) \in \{(1,1), (1,2), (2,1), (2,3), (3,2)\}$. As A_{33} is stable, there is a unique nonnegative definite solution P_{33} to the $(3, 3)$ equation. This follows from Lyapunov's Theorem in the version of Carlson and Schneider (see, e.g., [LT85, Theorem 13.1.3]). As P_{11} is zero, it follows that $P_{13} = 0 = P_{31}^*$ for any element of $\mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$. The only non-unique submatrix of any $\tilde{P} \in \mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$ partitioned as in (17) is P_{22}. As $P_{22} \geq 0$ for any $\tilde{P} \in \mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$ and $P_{22} = 0$ yields an element of $\mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$ it is clear that $\tilde{P} = \mathrm{diag}\,(0,\, 0,\, P_{33})$ is the unique minimal solution of the transformed equation (3).

As the ordering of Hermitian matrices is preserved under equivalence transformations, the minimal solution of the original equation is given by

$$P_{\mathcal{N}} := S^{-*}\mathrm{diag}\,(0,\, 0,\, P_{33})\, S^{-1}. \qquad ■ \tag{18}$$

The next lemma provides a characterization for the minimal solution which is obviously satisfied in Example 3.2.

Lemma 3.4 *If $P \in \mathcal{L}(A, C)$, then P is the minimal solution if and only if*

$$E_{\geq}(A) \subset \ker(P). \tag{19}$$

Proof. If $P \in \mathcal{L}(A, C)$ is the minimal solution, then (19) follows from (18).

On the other hand, as $P \in \mathcal{L}(A, C)$ it follows that $\ker(P) \subset \ker(C)$. This is easily seen when pre-multiplying (3) by x^* and post-multiplying by x for any $x \in \ker(P)$. Hence, from (19) we obtain $E_{\geq}(A) \subset \ker(C)$. Using the same transformations as in the proof of Lemma 3.1 such that A, C are as in (12), it follows that $C_1 = 0$ and $\tilde{P} = S^* P S \in \mathcal{L}(\tilde{A}, \tilde{C})$ has the form $\tilde{P} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & P_{22} \end{smallmatrix}\right]$ partitioned according to (12). By Lemma 2.1, P_{22} is nonnegative definite and uniquely defined because A_{22} is stable and furthermore, $\tilde{P} \in \mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$. Moreover, the representation (18) of the uniquely defined minimal solution yields that $\tilde{P}$ and hence P have to be the minimal elements of $\mathcal{L}_{\mathcal{N}}(\tilde{A}, \tilde{C})$ and $\mathcal{L}_{\mathcal{N}}(A, C)$, respectively. ■

The minimal solution is, by definition, nonnegative definite. Thus, the previous lemma provides a sufficient condition for a Hermitian solution to be nonnegative definite. This will prove useful later.

Corollary 3.5 *If $P \in \mathcal{L}(A, C)$ and $E_{\geq}(A) \subset \ker(P)$, then $P \geq 0$.*

To define the sequence $\{P_i\}_{i \geq 1}$ as in (4)–(7), let $F_0 \in \mathbb{C}^{m \times n}$ be the initial value and consider, for the first step, the Lyapunov equation

$$A_0^* P_1 + P_1 A_0 + C_0^* C_0 = 0 \tag{20}$$

where $A_0 = A - BF_0$ and $C_0^* C_0 = F_0^* F_0 + C^* C$. A nonnegative definite solution P_1 is to be found. Consequently, $\mathcal{L}_{\mathcal{N}}(A_0, C_0) \neq \emptyset$ is a consistency condition for the initial value F_0; i.e., in order to start the iteration, a matrix F_0 admitting a nonnegative definite solution of (20) must be available.

Note that because of Lemma 3.1, the existence of P_1 is equivalent to the existence of some matrix $F_0 \in \mathbb{C}^{m \times n}$ such that $E_{\geq}(A_0) \subset \ker(C_0)$. In other words,

$$\mathcal{L}_{\mathcal{N}}(A_0, C_0) \neq \emptyset \quad \Longleftrightarrow \quad E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C). \tag{21}$$

That this condition can be satisfied will be demonstrated again using Example 3.2.

Example 3.2 (continued) *Let $B = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $c = 0$. Then (A, B) is nonstabilizable, but (21) is satisfied using $F_0 = [1,\ 0]$.*

The following proposition proves that in case (21) is satisfied, the Lyapunov equation (4) admits a nonnegative definite solution for each i. That is, the existence of some solution for the first Lyapunov equation guarantees that the complete sequence is well-defined.

Proposition 3.6 *If $P_{i+1} \in \mathcal{L}_{\mathcal{N}}(A_i, C_i)$ exists for some $i \geq 0$, then $P_{i+2} \in \mathcal{L}_{\mathcal{N}}(A_{i+1}, C_{i+1})$ also exists, where A_{i+1} and C_{i+1} are defined as in (5)-(7).*

Proof. From the existence of $P_{i+1} \in \mathcal{L}_{\mathcal{N}}(A_i, C_i)$ it follows that F_{i+1}, A_{i+1}, and C_{i+1} can be defined by (5)-(7). Moreover, analogous to the proof of Theorem 2.4, $P_{i+1} \in \mathcal{L}_{\mathcal{N}}(A_{i+1}, \tilde{C}_{i+1})$ where $\tilde{C}_{i+1}$ is given by (8). Hence, using Lemma 3.1 we obtain that

$$E_{\geq}(A_{i+1}) \subset \ker(\tilde{C}_{i+1}). \tag{22}$$

As $\ker(\tilde{C}_{i+1}) \subset \ker(C_{i+1})$ by (8), the existence of $P_{i+2} \in \mathcal{L}_{\mathcal{N}}(A_{i+1}, C_{i+1})$ follows from Lemma 3.1. ■

Corollary 3.7 *If $E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C)$ for some $F_0 \in \mathbb{C}^{m \times n}$, then the sequence $\{P_i\}_{i \geq 1}$ resulting from (4)-(7), is well defined.*

Proof. The fact $E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C)$ guarantees the existence of $P_1 \in \mathcal{L}_{\mathcal{N}}(A_0, C_0)$ because of (21). Inductively applying Proposition 3.6 proves the assertion. ■

Next, we investigate the convergence properties of the sequence $\{P_i\}_{i \geq 1}$. Note that $P_i \geq 0$ for all $i \in \mathbb{N}$ and hence for convergence it is sufficient to prove that the sequence is non-increasing. As in the proof of Theorem 2.4*(iii)*, the sequence is non-increasing if and only if the difference between consecutive solutions, $D_i = P_i - P_{i+1}$, is nonnegative definite. Moreover, as $D_i \in \mathcal{L}(A_i, F_{i-1} - F_i)$, the following result is a consequence of Lemma 3.4 and Corollary 3.5.

Proposition 3.8 *The sequence $\{P_i\}_{i \geq 1}$ is non-increasing if for all $i \geq 0$, the matrix D_i is minimal in $\mathcal{L}(A_i, F_{i-1} - F_i)$. This condition is equivalent to*

$$E_{\geq}(A_i) \subset \ker(D_i).$$

The previous proposition gives a sufficient condition for the convergence of the iteration. Now it remains to show that the sequence $\{P_i\}_{i \geq 1}$ can be defined in such a way that this condition is satisfied. Note that in the ith iteration, $P_i \in \mathcal{L}_{\mathcal{N}}(A_{i-1}, C_{i-1})$ is given and P_{i+1} is chosen as *one* solution in $\mathcal{L}_{\mathcal{N}}(A_i, C_i)$. As Equation (4) can have several solutions and if $P_{i+1} \in \mathcal{L}_{\mathcal{N}}(A_i, C_i)$ is chosen arbitrarily, $D_i = P_i - P_{i+1}$ may not satisfy the condition in Proposition 3.8.

Working with D_i instead of P_{i+1}, that is defining $P_{i+1} = P_i - D_i$ with $D_i \in \mathcal{L}_{\mathcal{N}}(A_i, F_{i-1} - F_i)$ minimal, the sequence $\{P_i\}_{i \geq 1}$ is non-increasing due to Proposition 3.8. However, in this case it is not guaranteed per se that $P_{i+1} \geq 0$. Obviously, a necessary and sufficient condition for the existence of some solution $D_i \in \mathcal{L}_{\mathcal{N}}(A_i, F_{i-1} - F_i)$ such that the corresponding matrix $P_{i+1} = P_i - D_i$ be nonnegative definite is that the minimal solution $D_i^m \in \mathcal{L}_{\mathcal{N}}(A_i, F_{i-1} - F_i)$ satisfies $P_i - D_i^m \geq 0$. The next proposition shows that this is always true.

Proposition 3.9 *If $D_i^m \in \mathcal{L}_{\mathcal{N}}(A_i, F_{i-1} - F_i)$ is minimal and furthermore, $P_i \in \mathcal{L}_{\mathcal{N}}(A_{i-1}, C_{i-1})$, then $P_{i+1} = P_i - D_i^m \geq 0$.*

Proof. The existence of the minimal solutions $P_i^m \in \mathcal{L}_{\mathcal{N}}(A_i, C_i)$ and $P_{i+1}^m \in \mathcal{L}_{\mathcal{N}}(A_{i+1}, C_{i+1})$ follows from the existence of P_i and Proposition 3.6. Moreover, by (8) and (22),

$$E_{\geq}(A_i) \subset \ker(\tilde{C}_i) \subset \ker(F_{i-1} - F_i) \tag{23}$$

and hence $E_{\geq}(A_i) \subset E_{\geq}(A_{i-1})$. Thus, using Lemma 3.4,

$$E_{\geq}(A_i) = E_{\geq}(A_i) \cap E_{\geq}(A_{i-1}) \subset \ker(D_i^m) \cap \ker(P_i^m) \subset \ker(D_i^m - P_i^m).$$

This fact, together with $P_i^m - D_i^m \in \mathcal{L}_{\mathcal{N}}(A_i, C_i)$ proves, by Lemma 3.4, that $P_i^m - D_i^m = P_{i+1}^m$. Therefore, using Corollary 3.5 it follows that $P_i - D_i^m \geq P_{i+1}^m \geq 0$. ■

Note that Proposition 3.9 can not be obtained as a consequence of Proposition 3.8: in Proposition 3.9 D_i^m is assumed to be minimal whereas in Proposition 3.8, D_i is defined as $P_i - P_{i+1}$.

Now we can state the convergence result for the Kleinman iteration for computing nonnegative definite solutions of (1) in case of general, not necessarily stabilizable, systems. The notation in (5)–(7) is maintained in the following theorem.

Theorem 3.10 *If there exists some $F_0 \in \mathbb{C}^{m \times n}$ such that the Lyapunov equation*

$$A_0^* P_1 + P_1 A_0 + C_0^* C_0 = 0$$

has a solution $P_1^* = P_1 \geq 0$, and we define the sequence $\{P_i\}_{i \geq 1}$ via

$$P_{i+1} = P_i - D_i^m \ , \ i \geq 1,$$

where D_i^m is the minimal solution of the Lyapunov equation

$$A_i^* D_i + D_i A_i + (F_{i-1} - F_i)^* (F_{i-1} - F_i) = 0 \ , \ i \geq 1,$$

then the sequence $\{P_i\}_{i \geq 1}$ is well defined and converges to a nonnegative definite solution of (1).

Proof. For any $i \geq 1$ the existence of P_i implies

$$E_{\geq}(A_i) \subset \ker(F_{i-1} - F_i)$$

in analogy to (23). Therefore, the existence of D_i^m follows from Lemma 3.1 which permits us to construct P_{i+1}. As P_1 exists by hypothesis, the sequence is well defined by induction.

Now from $D_i^m \geq 0$ we get that the sequence $\{P_i\}_{i\geq 1}$ is non-increasing. Moreover, as a a consequence of Proposition 3.9 and $P_1 \geq 0$ it follows that $P_i \geq 0$ for all i. This shows that $\tilde{P} = \lim_{i\to\infty} P_i \geq 0$ exists.

Finally, taking limits in (4) yields that $\tilde{P}$ is a solution of (1) as P_i is a solution of (4). ■

Some remarks are in order.

Remark 3.11 (i) *According to Lemma 3.1, the assumption on F_0 in Theorem 3.10 can be changed to $E_{\geq}(A_0) \subset \ker(C_0)$.*

(ii) *Note that, for $i \geq 2$, we have $A_i = A_{i-1} + BB^*D_{i-1}$ and the Lyapunov equation in the ith iteration is equivalent to $A_i^*D_i + D_iA_i + D_{i-1}BB^*D_{i-1} = 0$. Thus, we can compute D_i directly from D_{i-1}.*

(iii) *The limit of the sequence $\{P_i\}_{i\geq 1}$ can be expressed as*

$$\tilde{P} = P_1 - \sum_{i\geq 1} D_i.$$

The next theorem provides a result analogous to Theorem 3.10 in terms of the usual Kleinman iteration. The minimality of D_i will be guaranteed by the minimality of both P_i and P_{i+1}. Again, the notation from (5)–(7) will be employed.

Theorem 3.12 (general Kleinman iteration) *If there exists an initial matrix $F_0 \in \mathbb{C}^{m\times n}$ such that*

$$E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C)$$

and P_{i+1} is the minimal solution of the Lyapunov equation

$$A_i^*P_{i+1} + P_{i+1}A_i + C_i^*C_i = 0$$

for all $i \geq 0$, then the sequence $\{P_i\}_{i\geq 1}$ is well defined and converges to a nonnegative definite solution $\tilde{P}$ of the ARE (1). Moreover,

$$P_1 \geq P_2 \geq \ldots P_i \geq P_{i+1} \geq \ldots \geq \tilde{P}.$$

Proof. The sequence $\{P_i\}_{i\geq 1}$ is well defined because of Proposition 3.6. Moreover, the minimality of P_i guarantees $E_{\geq}(A_i) \subset \ker(P_{i+1})$ for each i due to Lemma 3.4. Hence,

$$E_{\geq}(A_i) = E_{\geq}(A_i) \cap E_{\geq}(A_{i+1}) \subset \ker(P_{i+1}) \cap \ker(P_{i+2}) \subset \ker(D_{i+1}),$$

where the first equality is a consequence of the inclusion $E_{\geq}(A_i) \subset E_{\geq}(A_{i+1})$ obtained from (23). Using Lemma 3.4 it follows that $D_i = P_i - P_{i+1}$ is minimal in $\mathcal{L}_{\mathcal{N}}(A_i, F_{i-1} - F_i)$. From Proposition 3.8 we get that the sequence is non-increasing. As the sequence is bounded from below by zero,

$\tilde{P} = \lim_{i\to\infty} P_i$ exists and is a nonnegative definite solution of the ARE (1) by taking limits in (4). ∎

Note that the iteration in Theorem 3.10 is equivalent to the generalization of the classical Kleinman iteration given in Theorem 3.12. This is a consequence of the assertions

$$\begin{array}{rcl} P_i \text{ minimal}, D_i \text{ minimal} & \Longrightarrow & P_{i+1} \text{ minimal} \\ P_{i+1} \text{ minimal}, P_i \text{ minimal} & \Longrightarrow & D_i \text{ minimal} \end{array}$$

which are contained in the proofs of Proposition 3.9 and Theorem 3.12.

However, in the first step, Theorem 3.12 requires the minimality of P_1 which is not necessary for Theorem 3.10. This is the reason why Theorem 3.12 permits only the construction of a particular solution of (1), whereas using Theorem 3.10, all the nonnegative definite solutions of (1) can be approximated by different choices of P_1 and using Remark 3.11*(iii)*.

Example 3.13 *Let* $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & a \end{smallmatrix}\right]$ *for* $a > 0$, $a \neq \sqrt{2}$, $B = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $C = [\,1,\ 0\,]$. *Then* (A, B) *is nonstabilizable. With* $F_0 = [\,1,\ 0\,]$, *for the Kleinman iteration as given in Theorem 3.12 we obtain the sequence* $P_i := \left[\begin{smallmatrix} p_i & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $i = 1, 2, \ldots$, *where*

$$\{p_i\}_{i\geq 1} = \left\{ \frac{1}{2}, \frac{5}{12}, \frac{169}{408}, \frac{195025}{470832}, \frac{259717522849}{627013566048}, \ldots \right\},$$

The only positive semidefinite solution of the ARE is $\tilde{P} = \left[\begin{smallmatrix} -1+\sqrt{2} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. *Hence the sequence of relative errors obtained by the Kleinman iteration is*

$$\{\, 2.07 \times 10^{-1}, 5.92 \times 10^{-3}, 5.13 \times 10^{-6}, 3.85 \times 10^{-12}, 2.68 \times 10^{-16}, \ldots \}.$$

The last example implies quadratic convergence of the sequence generated by the Kleinman iteration. However, it is not yet clear how the conditions for the convergence rate of the iteration as investigated in [GL98] for the case that (A, B) is stabilizable transform to the situation considered here.

The Kleinman iteration as given in Theorem 3.12 can be used to compute special solutions in $\mathcal{R}_{\mathcal{N}}(A, B, C)$. Of particular interest is the minimal element of $\mathcal{R}_{\mathcal{N}}(A, B, C)$ as this is needed for the solution of the linear-quadratic optimal control problem without stability [Gee88].

Theorem 3.14 *Let the matrix triple* (A, B, C) *be in Kalman canonical form and assume the assumptions of Theorem 3.12 hold. Then choosing*

$$F_0 = \left[\begin{array}{cccc} 0 & F_2 & 0 & F_4 \end{array}\right]$$

such that $A_2 - B_2 F_2$ *is stable, the sequence* $\{P_i\}_{i\geq 1}$ *generated by the Kleinman iteration converges to the minimal solution of (1) in* $\mathcal{R}_{\mathcal{N}}(A, B, C)$.

Proof. From the assumed properties of F_0 we obtain that $E_{\geq}(A_0) \subset \ker(F_0) \cap \ker(C)$. By Theorem 3.12, the Kleinman iteration converges to a nonnegative definite solution. Denote this solution by P_*.

By Lemma 3.4 we get that $E_{\geq}(A_0) \subset \ker(P_1)$ by the minimality of P_1. Moreover, from the proof of Theorem 3.12 we now that the sequence $\{P_i\}_{i \geq 1}$ is non-increasing. Hence, $\ker(P_i) \subset \ker(P_{i+1}) \subset \ker(P_*)$. It follows that $E_{\geq}(A_0) \subset \ker(P)$.

Moreover, we have $V_{\geq}(A, C) := V(A, C) \cap E_{\geq}(A) \subset E_{\geq}(A_0)$ because for all $x \in V(A, C)$, x has the form $x = \begin{bmatrix} x_1 & 0 & x_3 & 0 \end{bmatrix}$ and by the choice of F_0 we obtain $Ax = A_0 x$.

Collecting these results yields $V_{\geq}(A, C) \subset \ker(P_*)$. As the minimal solution is the unique element of $\mathcal{R}_{\mathcal{N}}(A, B, C)$ satisfying $\ker(P) = V(A, C)$ (see [Gee88, Wim95]) and $V_{<}(A, C) := V(A, C) \cap E_{<}(A) \subset \ker(P)$ for all $P \in \mathcal{R}_{\mathcal{N}}(A, B, C)$ (see [Wim94]), it follows that $P_* \equiv P_{\min}$. ■

4 Existence of Nonnegative Definite Solutions of the ARE

Using Theorem 3.12 or Theorem 3.10 a necessary and sufficient condition for the existence of nonnegative definite solutions of (1) is obtained from the existence of some initial value $F_0 \in \mathbb{C}^{m \times n}$ satisfying

$$E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C).$$

At least two particular cases are evident. If the system (A, B) is stabilizable then, choosing F_0 such that A_0 is stable, we have $E_{\geq}(A_0) = \{0\} \subset \ker(F_0) \cap \ker(C)$. In case the unstable part of A is also unobservable, that is if $E_{\geq}(A) \subset \ker(C)$, then we can choose $F_0 = 0$. The next theorem shows that a consistent initial matrix F_0 exists if and only if $\mathcal{R}_{\mathcal{N}}(A, B, C) \neq \emptyset$.

Theorem 4.1 *The ARE (1) admits a nonnegative definite solution if and only if there exists a matrix $F_0 \in \mathbb{C}^{m \times n}$ satisfying*

$$E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C). \tag{24}$$

Proof. If $P \in \mathcal{R}_{\mathcal{N}}(A, B, C)$ then, choosing $F_0 = B^* P$, the ARE (1) can be written as

$$A_0^* P + P A_0 + C_0^* C_0 = 0, \quad A_0 = A - BF_0, \quad C_0^* C_0 = F_0^* F_0 + C^* C,$$

which admits P as a solution. Thus, from Lemma 3.1 we get

$$E_{\geq}(A - BF_0) \subset \ker(C_0) = \ker(F_0) \cap \ker(C).$$

Conversely, the existence of $F_0 \in \mathbb{C}^{m \times n}$ satisfying (24) together with Theorem 3.12 (or Theorem 3.10) proves the existence of

$$\tilde{P} = \lim_{i \to \infty} (P_i) \in \mathcal{R}_{\mathcal{N}}(A, B, C). \qquad ■$$

Remark 4.2 *The following assertions are equivalent to the existence of some solution $P \in \mathcal{R}_{\mathcal{N}}(A, B, C)$:*

1. $E_{\geq}(A) \subset V(A, C) + R(A, B)$.

2. $\{N \in \mathrm{Inv}(A) \mid V_{\leq}(A, C) \subset N \subset E_{\geq}(A) \subset V(A, C), N + R(A, B)\}$ *is not empty.*

3. $E_{>}(A) \subset V(A, C) + R(A, B)$ *and* $\dim(E_{=}(H)) = 2\dim(V_{=}(A, C))$, *where* $H = \left[\begin{smallmatrix} A & -BB^* \\ -C^*C & -A^* \end{smallmatrix}\right]$ *is the* Hamiltonian matrix *corresponding to (1).*

4. $E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C)$ *for some* $F_0 \in \mathbb{C}^{m \times n}$.

Condition 1 in Remark 4.2 is contained in [GH90] and Condition 2 can be found in [Wim94]. The equivalence between Conditions 1 and 2 is obvious, taking $N = V(A, C)$. Condition 3 can be found in [PH94] for the differential periodic Riccati equation and in [Wim94] for the ARE. A proof for the equivalence between Conditions 2 and 3 is given in [Wim94].

The equivalence between Condition 4 and the rest is a consequence of Theorem 4.1, but a direct proof of this equivalence is particularly interesting because it provides a geometric interpretation for the "consistency condition" which devises a way to obtain an initial matrix F_0. The direct proof of the equivalence between Conditions 1 and 4 corresponds to Proposition 4.3 below.

For the following, we may assume without loss of generality that the ARE (1) is given in this form. Obviously Condition 1 in Remark 4.2 is equivalent to the stability of the uncontrollable and observable part of (A, B, C), that is, to the stability of A_4 in the Kalman decomposition (2).

Proposition 4.3 *There exists a state feedback matrix $F_0 \in \mathbb{C}^{m \times n}$ such that $E_{\geq}(A - BF_0) \subset \ker(F_0) \cap \ker(C)$ if and only if in the Kalman canonical form of A as given in (2), A_4 is stable.*

Proof. Suppose that $F_0 = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix}$ is partitioned analogous to (2). Then the matrix $A_0 = A - BF_0$ has the form

$$A_0 = \begin{bmatrix} A_1 - B_1F_1 & A_{12} - B_1F_2 & A_{13} - B_1F_3 & A_{14} - B_1F_4 \\ -B_2F_1 & A_2 - B_2F_2 & -B_2F_3 & A_{24} - B_2F_4 \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{bmatrix}.$$

Thus, each eigenvalue λ of A_4 with nonnegative real part is also an eigenvalue of A_0. Hence, there exists some $x \in E_{\geq}(A_0)$ such that $x^T = \begin{bmatrix} x_1^T \, x_2^T \, x_3^T \, x_4^T \end{bmatrix}$ with $x_4 \neq 0$. Such a vector satisfies $x \in E_{\geq}(A_0)$ but $x \notin \ker(C) = V_1 \oplus V_3$.

Conversely, if A_4 is stable, choosing F_2 such that $A_2 - B_2F_2$ is stable and $F_0 = \begin{bmatrix} 0 & F_2 & 0 & 0 \end{bmatrix}$, the matrix $A_0 = A - BF_0$ is given by

$$A_0 = \begin{bmatrix} A_1 & A_{12} - B_1F_2 & A_{13} & A_{14} \\ 0 & A_2 - B_2F_2 & 0 & A_{24} \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{bmatrix}.$$

Then, each $x \in E_{\geq}(A_0)$ partitioned as above satisfies that $x_4 \in E_{\geq}(A_4) = \{0\}$ and $x_2 \in E_{\geq}(A_2 - B_2F_2) = \{0\}$. This implies $x \in V_1 \oplus V_3 \subset \ker(F_0) \cap \ker(C)$ which completes the proof. ■

The proof of Proposition 4.3 shows how a consistent initial matrix F_0 which is needed for starting the Kleinman iteration can be obtained: first, compute the Kalman decomposition of (A, B, C). Then stabilize the subsystem (A_2, B_2) by a state feedback matrix F_2. This can be achieved by some stabilization procedure as described, e.g., in [Sim96]. The initial matrix is then given as $F_0 = \begin{bmatrix} 0 & F_2 & 0 & 0 \end{bmatrix}$.

5 Concluding Remarks

We have shown that the Kleinman iteration (or Newton's method) for the ARE (1) may converge to a nonnegative solution of the ARE even if the underlying system is nonstabilizable. Necessary and sufficient condition for convergence of the Kleinman iteration to some nonnegative definite solution of the ARE are derived. These conditions together with the Kleinman iteration can be used to prove the existence of nonnegative solutions of the ARE. Using the Kalman decomposition of the underlying linear time-invariant system, we have also described a constructive way to compute an initial state feedback F_0 from which the Kleinman iteration converges to some nonnegative solution of the ARE. If the initial value is chosen in a special way, the iteration converges to the minimal nonnegative definite solution of the ARE. This can be used in order to solve the linear-quadratic optimal control problem without stability.

The realization of the methods derived in this paper as efficient numerical algorithms requires further study. Most of the results are based on the Kalman decomposition of the underlying linear time-invariant system. Computing this decomposition is relatively expensive and is based on crucial rank decisions. Therefore it will be more appropriate to work with the original data without going to the Kalman decomposition. The initialization of the Kleinman iteration as well as the numerical solution of the Lyapunov equation in each iteration for the minimal solution without working with the Kalman decomposition are open problems and future work is needed here.

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