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Two-mechanism models with plastic mechanisms - modelling in continuum-mechanical framework

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Abstract

This note deals with two-mechanism models (= 2M models). 2M models (or, generally, multi-mechanism models) are a useful tool for modelling of complex material behavior. They have been studied and applied for the last twenty years. This note is focused on the modelling of 2M models in continuum-mechanical framework mainly in the case of plastic behaviour. In particular, some classes of 2M models are described, thermodynamic consistency of some 2M models is proved, coupled evolution equations for the backstresses are derived. Moreover, an extension concerning coupling between isotropic and kinematic hardening is proposed. Finally, some mathematical problems arising from 2M models are formulated.

1 Introduction

1) Two-mechanism (or, generally, multi-mechanism) models have been studied and applied for the last twenty years. Their characteristic trait is the additive decomposition of the inelastic (i.e., plastic or visco-plastic, e.g.) strain into two (or multi) parts (sometimes called "mechanisms") in the case of small deformations. In comparison with rheological models (cf. Palmov (1998), e.g.), there is an interaction between these mechanisms (see Figure 1). This interaction allows to describe important observable effects, but, it requires additional efforts in modelling and simulation. Each inelastic strain part may exhibit plastic, or general inelastic behavior. The (thermo-)elastic strain is not regarded as an own mechanism. Each mechanism has its own internal variables with corresponding evolution equations. Moreover, each mechanism may have an own yield criterion, or, there may be common yield criteria for several mechanisms. Thus, in the case of two mechanisms, there are possible models of the type 2M1C and 2M2C. That means two mechanisms with one or two yield criteria (see Figure 2). A mechanism without yield criterion like creep can be formally treated as a mechanism with its own criterion with zero yield stress.

If the inelastic strain is seen as one mechanism (as it was historically first), one refers to about a "unified model" (or Chaboche model) (cf. the survey by Chaboche (2008) and the references cited therein). (That means plastic and viscous components are considered together in the same variable.) As explained in Contesti and Cailletaud (1989) and Cailletaud and Saï (1995), there are experimentally observable effects (inverse strain-rate sensibility, e.g.) which can be qualitatively correctly described by the two-mechanism approach.

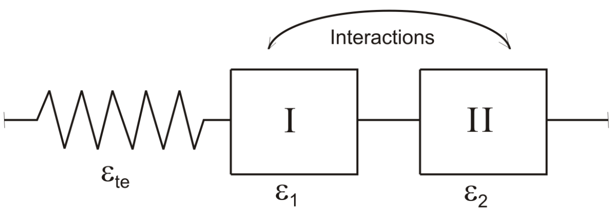

Figure 1: Scheme of a two-mechanism model. The two inelastic mechanisms 1 and 2 have their own evolution equations. But, they are not independent from each other. The thermoelastic strain $\boldsymbol{\varepsilon}_{te}$ is usually not regarded as a mechanism.

2) Up to now, there are only relatively few publications dealing directly with multi-mechanism models. We refer to Contesti and Cailletaud (1989), Saï (1993), Cailletaud and Saï (1993), Cailletaud and Saï (1995), Blaj and Cailletaud (2000), Besson et al. (2001), Saï et al. (2004), Aeby-Gautier and Cailletaud (2004), Taleb et al. (2006), Velay et al. (2006), Saï and Cailletaud (2007), Wolff and Taleb (2008), Chaboche (2008), Hassan et al. (2008), Taleb and Hauet (2009), Taleb and Cailletaud (2010), Wolff et al. (2010a). Finally, there is a recent survey by Saï (2010).[1]

In contrary to this manageable number, there is a large variety of papers dealing with complex material behavior of metals, soils, composites, biological tissues etc. in which the inelastic strain is decomposed into several parts. But, as a rule, multi-mechanism models are not directly addressed. We give some examples below.

To our knowledge, a first systematic formulation and investigation of two mechanism models was given by Contesti and Cailletaud (1989). Besides, the papers by Cailletaud and Saï (1995), by Saï and Cailletaud (2007), by Taleb and Cailletaud (2010) and, by Saï (2010) give overviews and possibilities of

[1] The paper by Saï (2010) has been directly published before ending our study.

applications. Moreover, we refer to the thesis of Saï (1993) and to the book by Besson et al. (2001). The survey article by Chaboche (2008) contains comments concerning multi-mechanism models, too.

Wolff and Taleb (2008) proved thermodynamic consistency of two-mechanism models dealt with in Taleb et al. (2006). The question about thermodynamic consistency is not trivial, if one leaves the class of "generalized standard models" (cf. Besson et al. (2001), e.g.). This is the case for important model modifications (cf. Taleb et al. (2006), Saï and Cailletaud (2007)). Additionally, there is the typical *mutual influence of mechanisms* (in particular via the backstresses). Thus, generally, a separate investigation of thermodynamic consistency with respect to each mechanism is not successful. This is a substantial difference to rheologic models (cf. Palmov (1998), e.g.).

Generally, the material parameters depend on temperature. Most of the papers about multi-mechanism models cited above only consider the isothermal case, as ratcheting experiments, up to now, are only performed under constant temperature. In the current paper we will also address the *non-isothermal* case. This leads to more complex equations at some places.

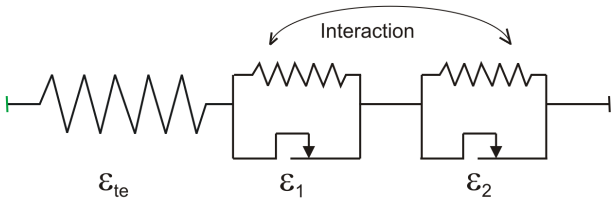

Figure 2: 2M2C model with two plastic mechanisms with kinematic hardening.

3) An important application of two-mechanism models is cyclic plasticity including ratcheting. There are many papers dealing with ratcheting both in modelling as well as in simulation and comparison with experimental data. For general modelling and simulation we exemplarily refer to Portier et al. (2000), Bari and Hassan (2002), Taleb et al. (2006), Kang (2008), Jiang and Zhang (2008), Hassan et al. (2008), Abdel-Karim (2009), Taleb and Hauet (2009), Krishna et al. (2009), Abdel-Karim (2010) and the references therein.

In the majority of the literature ratcheting is dealt within the framework of one-mechanism models. Investigations of ratcheting with the aid of two-mechanism models can be found in Cailletaud and Saï (1995), Blaj and Cailletaud (2000), Saï et al. (2004) [using a 2M2C model], Taleb et al. (2006), Velay et al. (2006), Saï and Cailletaud (2007), Hassan et al. (2008), Taleb and Hauet (2009), Taleb and Cailletaud (2010).

Finally, experiments and simulations must decide, in which situation which model delivers the better approximation of the reality. In Hassan et al. (2008), a direct comparison between a modified Chaboche model and a 2M model has been performed.

4) Another important application of two-mechanism models lies in modelling of complex material behavior of steel under phase transformations. The two-mechanism approach directly used in Videau et al. (1994) and Wolff et al. (2008) allows a good description of interactions between classical and transformation-induced plasticity. On the other hand, in Leblond et al. (1986a), Leblond et al. (1986b), Leblond et al. (1989), Leblond (1989), Fischer et al. (1998), Fischer et al. (2000), Devaux et al. (2000), Taleb and Sidoroff (2003), the transformation-induced plasticity itself is the focus, and the two-mechanism approach arises in natural way without a special reference.

More recent experiments and simulations (cf. Taleb and Petit (2006), e.g.) show that, in some cases, the transformation-induced plasticity after a pre-deformation of austenite *cannot* be qualitatively correctly described with the aid of the model developed in Leblond et al. (1986a), Leblond et al. (1986b), Leblond et al. (1989), Leblond (1989), Devaux et al. (2000), Taleb and Sidoroff (2003). However, the consistent access via the two-mechanism model allows a qualitatively correct description of this phenomenon (cf. Wolff et al. (2008), Wolff et al. (2009)). Based on this approach, in Suhr (2010) and Wolff et al. (2010b), a semi-implicit algorithm for numerical simulations has been developed, and some simulations are presented.

Contrary to Videau et al. (1994), Wolff et al. (2008), Mahnken et al. (2009) and others, in Aeby-Gautier and Cailletaud (2004) the material behavior of steel is described by a multi-mechanism model at the macro level as well as at the meso level (sometimes called micro level), whereas the proof of thermodynamic consistency still remains open. Furthermore, it should be noted that some authors combine classical and transformation-induced plasticity in one model ("unified transformation-thermoplasticity", cf. Inoue and Tanaka (2006)).

5) The complex material behavior of important materials (such as visco-plastic materials, shape-memory alloys, soils, granular materials, composites, biological tissues) leads to multi-mechanism models, when taking the additive decomposition of the strain tensor into account. However, in most cases, the concrete application is *is not* set in the framework of multi-mechanism models in the sense of Cailletaud and Saï (1995). We give some examples.

When modelling shape-memory alloys, sometimes, the inelastic part of the strain tensor is decomposed into two parts (into two summands in the case of small deformations). We refer to Helm and Haupt (2003), Helm (2007), Reese and Christ (2008), Kang et al. (2009), Kan and Kang (2010) e.g.

The material behavior of salt in deposits is very complex, and its modelling uses an additive decomposition of inelastic strain into three parts (cf. Munson et al. (1993), e.g.). In Chan et al. (1994), Koteras and Munson (1996), an additional summand is used which is induced by damage.

Further references to modelling via several mechanisms can be found in some papers in geomechanics, for instance, for cohesionless soil in Shi and Xie (2002), for clay in Modaressi and L. (1997), for sand in Akiyoshi et al. (1994), Fang (2003) and for granular material in Anandarajah (2008). Similarly, complex material behavior of biologic tissue is modelled using a multi-mechanism approach (cf. Wulandana and Robertson (2005), Doehring et al. (2004), e.g.).

6) This work is organised as follows:

- In Sections 2, 3, and 4, some classes of 2M models are described in the thermodynamical framework.
- In Sections 3, 4 some results on thermodynamic consistency are presented.
- In Section 5, we present useful relations generalizing the classical Armstrong-Frederick equations for backstresses.
- In Section 6, an extension concerning coupling between kinematic and isotropic hardening is given.
- In Section 7, some resulting mathematical problems are formulated in short.

Basically, the material in Sections 2, 3, 4, and 5 can be found in Wolff et al. (2010a). Generally, the material in Section 5 is known. However, here, it is comprehensively presented. The topic in Subsection 5.4.2 is based on investigations in Taleb et al. (2006). Section 7 gives an outlook for further mathematical investigations, including simulations.

In this note, we focus on 2M models with plastic mechanisms. But, generally, 2M models with viscoplastic, creep or more complex mechanisms can be dealt with in an analogous manner (cf. Remark 3.2).

2 Some basic facts on two-mechanism models

In this section we provide important basic relations for 2M models. At first, there will be common items for models with one and with two yield criteria. After this, we deal separately with 2M models with one and with two criteria.

2.1 General assertions

We restrict ourselves to small deformations. Thus, the equation of momentum, the energy equation and the Clausius-Duhem inequality are given by

$$\varrho\, \ddot{\boldsymbol{u}} - \operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \tag{2.1}$$

$$\varrho\, \dot{e} + \operatorname{div} \boldsymbol{q} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r \tag{2.2}$$

$$-\varrho\, \dot{\psi} - \varrho\, \eta\, \dot{\theta} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{\theta}\, \boldsymbol{q} \cdot \boldsymbol{\nabla}\theta \geq 0. \tag{2.3}$$

The relations (2.1) - (2.3) have to be fulfilled in the space-time domain $\Omega\times]0,T[$. The notation is standard: ϱ - density in the reference configuration, that means for $t=0$, $\boldsymbol{u}$ - displacement vector, $\boldsymbol{\varepsilon}$ - linearized

Green strain tensor, θ - absolute temperature, $\boldsymbol{\sigma}$ - Cauchy stress tensor, $\boldsymbol{f}$ - volume density of external forces, e - mass density of the internal energy, $\boldsymbol{q}$ - heat-flux density vector, r - volume density of heat supply, ψ - mass density of free (or Helmholtz) energy, η - mass density of entropy. The time derivative is denoted by a dot. $\boldsymbol{\alpha} : \boldsymbol{\beta}$ is the scalar product of the tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, $\boldsymbol{q} \cdot \boldsymbol{p}$ is the scalar product of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$. We note the well-known relations

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T), \quad \psi = e - \theta\, \eta. \tag{2.4}$$

In the general case of inelastic material behavior, the full strain $\boldsymbol{\varepsilon}$ is split up via

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{te} + \boldsymbol{\varepsilon}_{in} \tag{2.5}$$

($\boldsymbol{\varepsilon}_{te}$ - thermoelastic strain, $\boldsymbol{\varepsilon}_{in}$ - inelastic strain). Usually, the inelastic strain is assumed to be traceless, i.e.

$$\mathrm{tr}(\boldsymbol{\varepsilon}_{in}) = 0. \tag{2.6}$$

The accumulated inelastic strain is defined by

$$s_{in}(t) := \int_0^t (\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_{in}(\tau) : \dot{\boldsymbol{\varepsilon}}_{in}(\tau))^{\frac{1}{2}}\ d\tau. \tag{2.7}$$

We drop the dependence on the space variable x.

We propose for the free energy ψ the split

$$\psi = \psi_{te} + \psi_{in}. \tag{2.8}$$

The thermoelastic part is given by

$$\psi_{te} := \frac{1}{\varrho}\{\mu\ \boldsymbol{\varepsilon}^*_{te} : \boldsymbol{\varepsilon}^*_{te} + \frac{K}{2}(\mathrm{tr}(\boldsymbol{\varepsilon}_{te}))^2 - 3\,K\,\alpha(\theta - \theta_0)\,\mathrm{tr}(\boldsymbol{\varepsilon}_{te}) + C(\theta - \theta_0)\}. \tag{2.9}$$

$\mu > 0$ - shear modulus, $K > 0$ - compression modulus, α - linear heat-dilatation coefficient, θ_0 - initial temperature, i.e. for $t = 0$, C - calorimetric function (cf. Helm and Haupt (2003), e.g.), $\boldsymbol{\varepsilon}^*_{te}$ - deviator of $\boldsymbol{\varepsilon}_{te}$, defined (in $3d$ case) by

$$\boldsymbol{\varepsilon}^*_{te} = \boldsymbol{\varepsilon}_{te} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon}_{te})\boldsymbol{I} \qquad (\boldsymbol{I} \text{ - unity tensor}). \tag{2.10}$$

We assume that the inelastic part ψ_{in} of ψ has the general form

$$\psi_{in} = \psi_{in}(\xi,\ \theta). \tag{2.11}$$

$\xi = (\xi_1, \ldots, \xi_m)$ (ξ_j - scalars or tensors) represent the internal variables. Further on, these variables will be chosen in accordance with concrete models under consideration. Internal variables are used for description of phenomena like

- inelastic deformations, including
  - dislocation-based plasticity and viscoplasticity with kinematic and isotropic hardening as well as with softening
  - transformation-induced plasticity in steels
  - specific inelastic deformations in shape-memory materials
- phase transformations
- damage
- changes in microstructure, in polymers, e.g. (cf. Lion and Höfer (2007))

In the case of damage, the thermoelastic part ψ_{te} of the free energy may depend on internal variables too (cf. Besson et al. (2001)). Internal variables have to fulfil evolution equations which are usually ordinary differential equations (ODE) with respect to the time t. As a rule, one poses zero initial conditions, i.e.

(2.12) $$\xi_j(0) = 0 \qquad \text{for } j = 1, \dots, m.$$

Taking (2.8), (2.9) and the general relation in thermodynamics

(2.13) $$\boldsymbol{\sigma} = \varrho \, \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}_{te}}$$

into account, one obtains the usual material law in thermo-elasto-plasticity:

(2.14) $$\boldsymbol{\sigma} = 2\mu \, \boldsymbol{\varepsilon}_{te}^{*} + K\mathrm{tr}(\boldsymbol{\varepsilon}_{te})\boldsymbol{I} - 3\,K\,\alpha(\theta - \theta_0)\boldsymbol{I}.$$

For materials with phase transformations, this relation (as well as (2.9)) will be extended (cf. Wolff et al. (2008))

Using standard arguments of thermodynamics (cf. Lemaitre and Chaboche (1990), Maugin (1992), Besson et al. (2001), Haupt (2002), e.g.), from (2.3) one obtains the dissipation inequality (as a reduced Clausius-Duhem inequality):

(2.15) $$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \varrho \sum_{j=1}^{m} \frac{\partial \psi_{in}}{\partial \xi_j} \dot{\xi}_j - \frac{1}{\theta} \, \boldsymbol{q} \cdot \boldsymbol{\nabla}\theta \geq 0$$

As common, a model is regarded as thermodynamically consistent, if the dissipation inequality (2.15) holds for all possible processes. Sufficient for this is, that (2.15) is fulfilled for arbitrarily chosen sets of its variables. As usual, we assume the Fourier law of heat conduction

(2.16) $$\boldsymbol{q} = -\kappa \, \boldsymbol{\nabla}\theta,$$

$\kappa > 0$ - heat conductivity (scalar or positive definite tensor). Therefore, the heat-conduction inequality

(2.17) $$-\frac{1}{\theta} \, \boldsymbol{q} \cdot \boldsymbol{\nabla}\theta \geq 0$$

is always fulfilled. Hence, the model under consideration is thermodynamically consistent, if the **remaining inequality**

(2.18) $$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \varrho \sum_{j=1}^{m} \frac{\partial \psi_{in}}{\partial \xi_j} : \dot{\xi}_j \geq 0$$

is fulfilled.

By (2.2) and standard arguments (cf. Besson et al. (2001), Haupt (2002), e.g.) one obtains the heat-conduction equation:

(2.19) $$\varrho\, c_d\, \dot{\theta} - \mathrm{div}(\kappa\, \boldsymbol{\nabla}\theta) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{in} - \varrho \sum_{j=1}^{m} \frac{\partial \psi_{in}}{\partial \xi_j} : \dot{\xi}_j + \varrho\, \theta \sum_{j=1}^{m} \frac{\partial^2 \psi_{in}}{\partial \theta \, \partial \xi_j} \dot{\xi}_j + \theta \, \frac{\partial \boldsymbol{\sigma}}{\partial \theta} : \dot{\boldsymbol{\varepsilon}}_{te} + r$$

($c_d = \theta \frac{\partial^2 \psi}{\partial \theta^2} > 0$ - specific heat).

In the theory of 2M models the following decomposition is crucial:

(2.20) $$\boldsymbol{\varepsilon}_{in} = A_1\, \boldsymbol{\varepsilon}_1 + A_2\, \boldsymbol{\varepsilon}_2,$$

A_1, A_2 are positive real numbers. As usual, the inelastic strains are trace-less:

(2.21) $$\mathrm{tr}(\boldsymbol{\varepsilon}_{in}) = \mathrm{tr}(\boldsymbol{\varepsilon}_1) = \mathrm{tr}(\boldsymbol{\varepsilon}_2) = 0.$$

Remark 2.1. (i) The parameters A_1 and A_2 open opportunities for further extensions and special applications. We refer to Saï and Cailletaud (2007). A_1 and A_2 can depend on further quantities as, for instance, they can constitute phase fraction in complex materials (steel, shape memory alloys, e.g.). In this sense, here is a bridge from the macro to the meso (or micro) level of modelling.
(ii) In case of n mechanisms, instead of (2.20), one has the split

(2.22) $$\boldsymbol{\varepsilon}_{in} = \sum_{j=1}^{n} A_j\, \boldsymbol{\varepsilon}_j$$

with $A_j > 0$.

For both $\boldsymbol{\varepsilon}_j$ we introduce *separate* accumulations

$$s_j(t) := \int_0^t (\frac{2}{3}\,\dot{\boldsymbol{\varepsilon}}_j(\tau) : \dot{\boldsymbol{\varepsilon}}_j(\tau))^{\frac{1}{2}}\; d\tau \quad j = 1, 2. \tag{2.23}$$

Note, that s_{in} (as defined in (2.7)) is not the sum of s_1 and s_2. As the roots in (2.7) and (2.23) are norms, one gets useful inequalities

$$|A_1\,\dot{s}_1 - A_2\,\dot{s}_2| \le \dot{s}_{in} \le A_1\,\dot{s}_1 + A_2\,\dot{s}_2. \tag{2.24}$$

We introduce the local stresses $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ via

$$\boldsymbol{\sigma}_j := A_j\,\boldsymbol{\sigma} \qquad\qquad j = 1, 2 \tag{2.25}$$

From now on, we deal separately with 2M1C and 2M2C models. In order to focus, we preferably deal with plastic mechanisms obeying yield criteria. Other types of (inelastic) mechanisms can be dealt with analogously. As mentioned in the Introduction, a mechanism without yield stress can be formally treated as a mechanism with zero yield stress.

2.2 Two-mechanism models with one yield criterion

We specialize the ansatz for the inelastic part of the free energy in (2.11), focussing on plastic mechanisms, and assuming the internal variables to be given $\xi = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q)$. This leads to

$$\psi_{in} = \psi_{in}(\theta, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q). \tag{2.26}$$

The tensorial symmetric internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are related to kinematic hardening, the scalar internal variable q is related to isotropic hardening. All of them are of strain type. $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are associated with the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$, respectively. We define the backstresses $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ associated with the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$, respectively, as well as the isotropic hardening R by

$$\boldsymbol{X}_j = \varrho \frac{\partial \psi_{in}}{\partial \boldsymbol{\alpha}_j} \qquad (j = 1, 2), \tag{2.27}$$

$$R = \varrho \frac{\partial \psi_{in}}{\partial q}. \tag{2.28}$$

(2.25), (2.27), (2.28) and (2.18) imply the following remaining inequality

$$(\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) - R\,\dot{q} \ge 0, \tag{2.29}$$

and, via (2.19)

$$\begin{aligned} \varrho\, c_d\, \dot{\theta} - \operatorname{div}(\kappa\, \nabla\theta) =& (\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) + \\ & - R\,\dot{q} + \theta\,\frac{\partial \boldsymbol{X}_1}{\partial\theta} : \dot{\boldsymbol{\alpha}}_1 + \theta\,\frac{\partial \boldsymbol{X}_2}{\partial\theta} : \dot{\boldsymbol{\alpha}}_2 + \theta\,\frac{\partial R}{\partial\theta}\,\dot{q} + \theta\,\frac{\partial \boldsymbol{\sigma}}{\partial\theta} : \dot{\boldsymbol{\varepsilon}}_{te} + r. \end{aligned} \tag{2.30}$$

Based on the von Mises stress, we define the quantities

$$J_j := (\frac{3}{2}(\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*) : (\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*))^{\frac{1}{2}} \qquad (j = 1, 2) \tag{2.31}$$

$$J := (J_1^N + J_2^N)^{\frac{1}{N}}. \tag{2.32}$$

The material parameter N has to fulfil

$$N \ge 1. \tag{2.33}$$

The yield function is given by

$$f := J - (R + R_0), \tag{2.34}$$

$$R_0 := \sqrt[N]{2}\sigma_0. \tag{2.35}$$

The initial yield stress $\sigma_0 = \sigma_0(\theta)$ can be determined by a standard tension experiment. Since we are dealing with plastic behavior, we suppose for all 2M1C models the subsequent constraint

$$f(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) \leq 0. \tag{2.36}$$

Clearly, the condition $f(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) < 0$ describes the elastic domain, and plastic deformation can only occur, if $f(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) = 0$.

Based on (2.31), (2.32), (2.34), we define

$$\mathbf{n}_j := -\frac{\partial f}{\partial \boldsymbol{X}_j} = \frac{3}{2}\frac{\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*}{J_j}\left(\frac{J_j}{J}\right)^{N-1} \qquad (j = 1, 2). \tag{2.37}$$

Remark 2.2. The importance of the parameter N in (2.33) for application consists in the fact, that, if it growths, the two quantities J_1 and J_2 become more and more independent of each other. We refer to Wolff and Taleb (2008), Taleb and Cailletaud (2010) for details.

2.3 Two-mechanism models with two yield criteria

Contrary to (2.26), now the inelastic free energy is given by

$$\psi_{in} = \psi_{in}(\theta, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q_1, q_2) \tag{2.38}$$

with $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ as above. q_1 and q_2 are related to the isotropic hardening of the first and second mechanism, respectively. The backstresses $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are defined as in (2.27), the isotropic hardenings R_1 and R_2 are defined by

$$R_j = \varrho\,\frac{\partial \psi_{in}}{\partial q_j} \qquad (j = 1, 2) \tag{2.39}$$

By (2.25), (2.27), (2.28) and (2.18) we infer

$$(\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) - R_1\,\dot{q}_1 - R_2\,\dot{q}_2 \geq 0. \tag{2.40}$$

Similarly as above, we obtain the specialized heat-conduction equation

$$\begin{aligned} \varrho\, c_d\, \dot{\theta} - \operatorname{div}(\kappa\,\nabla\theta) &= (\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) + \\ &\quad - R_1\,\dot{q}_1 - R_2\,\dot{q}_2 + \theta\,\frac{\partial \boldsymbol{X}_1}{\partial\theta} : \dot{\boldsymbol{\alpha}}_1 + \theta\,\frac{\partial \boldsymbol{X}_2}{\partial\theta} : \dot{\boldsymbol{\alpha}}_2 + \theta\,\frac{\partial R_1}{\partial\theta}\dot{q}_1 + \theta\,\frac{\partial R_2}{\partial\theta}\dot{q}_2 + \theta\,\frac{\partial\boldsymbol{\sigma}}{\partial\theta} : \dot{\boldsymbol{\varepsilon}}_{te} + r. \end{aligned} \tag{2.41}$$

Now, the two yield functions are

$$f_j := J_j - (R_j + R_{0j}) \qquad j = 1, 2, \qquad (J_j \text{ defined by (2.31)}). \tag{2.42}$$

R_{0j} is the initial yield stress of the j^{th} mechanism. For all 2M2C models (in case of plastic mechanisms), the subsequent constraints are supposed

$$f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_{0j}) \leq 0 \qquad j = 1, 2. \tag{2.43}$$

And, as a consequence, a plastic deformation due to the j^{th} mechanism can only occur, if the j^{th} yield criterion is fulfilled, i.e. if

$$f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_{0j}) = 0. \tag{2.44}$$

Now, there are two elastic domains defined by $f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_{0j}) < 0$.

Finally, based on (2.31) and (2.42), for 2M2C models we define

$$\boldsymbol{n}_j := -\frac{\partial f_j}{\partial \boldsymbol{X}_j} = \frac{3}{2}\frac{\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*}{J_j}. \tag{2.45}$$

2.4 General remarks concerning the modelling

In this paragraph, we expose the further approach we follow.
(i) In order to obtain complete 2M models, one has to

- propose a concrete expression for the inelastic part of the free energy ψ_{in} (cf. (2.26) and (2.38)),
- to formulate evolutions laws
  - for the inelastic strains $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ as well as
  - for the internal variables $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and q (or q_1 and q_2).

Doing so, one has to take care that

- the inelastic free energy ψ_{in} is convex (with respect to the variables describing hardening) for frozen temperature (cf. remark 3.1),
- the remaining inequality is fulfilled for all admissible arguments. Thus, the model under consideration is thermodynamically consistent.

(ii) Moreover, the chosen model should approximate the reality as well as possible. This is an requirement of very great practical relevance, but it does not concern the theory itself.
(iii) We do *not* use dissipation potentials for modelling. This way opens more possibilities, as the evolution equations may be chosen independently of each other. We refer to Cailletaud and Saï (1995), Besson et al. (2001), Saï and Cailletaud (2007), Saï (2010) for using dissipation potentials in modelling of 2M models.
(iv) All arising material parameters (or more precisely material functions) may depend on temperature. Moreover, those parameters which do *not* occur in the free energy may additionally depend on stress and further quantities. This last dependency gives more possibilities for modelling, and, it does not influence the thermodynamic consistency.
(v) In case of more than two mechanisms, there are more possibilities of yield criteria. For instance, in a 3M model, there may be three separate criteria, or one common for all mechanisms, or one mechanism with a separate criterion, while the remaining two mechanisms have a common criterion.

3 Description and thermodynamic consistency of some 2M1C models

Now, we discuss two types of 2M1C models. Again, we collect the common things at the beginning.

3.1 Common features of 2M1C models

The inelastic free energy in (2.26) is further specialized as

(3.1) $$\psi_{in}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q, \theta) := \frac{1}{3\varrho}\left\{c_{11}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2\,c_{12}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + c_{22}(\theta)\,\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2\right\} + \frac{1}{2\varrho}Q(\theta)\,q^2,$$

Remark 3.1. **(i)** For each temperature, the inelastic free energy ψ_{in} in (3.1) is a convex function with respect to $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and q, if there hold (for all admissible θ) the conditions

(3.2) $$c_{11}(\theta) \geq 0, \qquad c_{12}(\theta)^2 \leq c_{11}(\theta)\,c_{22}(\theta),$$

(3.3) $$Q(\theta) \geq 0.$$

We note that the quadratic form related to c_{ij} is also positive semi-definite (cf. Wolff and Taleb (2008), e.g.). From the physical point of view, it is more precise to require that this part of the free energy is convex.
(ii) In order to focus, in this Section, we do not consider a possible coupling between kinematic and isotropic hardening in (3.1). This type of coupling will be considered in Section 6.

Assuming additionally

(3.4) $$c_{11} > 0 \qquad c_{22} > 0 \qquad Q > 0,$$

we exclude simpler cases.

Due to (2.27), (2.28) and (3.1), the following equations for $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and R hold

$$\boldsymbol{X}_1 = \frac{2}{3}\, c_{11}\, \boldsymbol{\alpha}_1 + \frac{2}{3}\, c_{12}\, \boldsymbol{\alpha}_2, \qquad \boldsymbol{X}_2 = \frac{2}{3}\, c_{12}\, \boldsymbol{\alpha}_1 + \frac{2}{3}\, c_{22}\, \boldsymbol{\alpha}_2, \tag{3.5}$$

$$R = Q\, q. \tag{3.6}$$

We assume evolution laws for the mechanisms $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$:

$$\dot{\boldsymbol{\varepsilon}}_j = \lambda\, \mathbf{n}_j \tag{3.7}$$

($\mathbf{n}_j$ - defined by (2.37)). The common plastic multiplier for both mechanisms $\lambda \geq 0$ has to fulfil

$$\lambda = 0, \quad \text{if} \quad f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) < 0, \tag{3.8}$$

$$\lambda \geq 0, \quad \text{if} \quad f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) = 0 \qquad \text{(flow condition)}. \tag{3.9}$$

Moreover, we assume an evolution law for q:

$$\dot{q} = r\, \lambda - \frac{b}{Q}\, R\, \lambda, \tag{3.10}$$

with r and b fulfilling

$$r > 0, \qquad b > 0, \tag{3.11}$$

($b = 0$ corresponds to the simpler case of linear isotropic hardening.) From (2.23), (2.31), (2.32), (2.37) and (3.7) one gets

$$\dot{s}_j = \lambda\, (J_1^N + J_2^N)^{\frac{1}{N}-1} J_j^{N-1}, \tag{3.12}$$

and, after this,

$$\lambda = ((\dot{s}_1)^{\frac{N}{N-1}} + (\dot{s}_2)^{\frac{N}{N-1}})^{\frac{N-1}{N}}. \tag{3.13}$$

We denote by Λ the primitive of λ, i.e.

$$\Lambda(t) = \int_0^t \lambda(\tau) d\tau. \tag{3.14}$$

There remain the approach for the evolution equation for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.We will discuss two variants leading to 2M models which are named here by 2M1C-a and 2M1C-b.

Remark 3.2. Viscoplastic mechanisms can be dealt with analogously without difficulties. Let be f as in (2.34) and $\mathbf{n}$ as in (2.37). Here, we consider 1C models. 2C models with viscoplastic mechanisms can be dealt with in the same manner. Formally, the evolution law for $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ looks like (3.7). Contrary to the plastic case, there is no constraint as in (2.36). The elastic domain is defined by

$$f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) \leq 0. \tag{3.15}$$

In general, the stress is not a-priori bounded. Hence, the viscoplastic multiplier is not determined by the flow condition, but it must be defined separately, for instance by

$$\lambda := \frac{2}{3\,\eta} < \frac{1}{D} f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) >^n . \tag{3.16}$$

The McCauley brackets $< \bullet >$ are defined by $< x >:= x$ for $x \geq 0$ and $< x >:= 0$ otherwise. The exponent $n > 0$ and the viscosity $\eta > 0$ generally depend on temperature (and maybe on other quantities). The drag stress (cf. Chaboche, 2008) is a positive scalar generally following its own evolution. Finally, there hold the relations (3.12) and (3.13) hold for λ and s_1, s_2.

Other inelastic mechanisms like creep can be dealt with in the same manner, if the rate of the corresponding inelastic strain is explicitly given.

3.2 The model 2M1C-a

We assume (3.1) - (3.4) and (3.7) - (3.11). In Addition, the evolution of $\dot{\boldsymbol{\alpha}}_j$ is given by

$$\dot{\boldsymbol{\alpha}}_j = a_j\,\dot{\boldsymbol{\varepsilon}}_j - \frac{3d_j}{2c_{jj}}\,\{(1-\eta_j)\,\boldsymbol{X}_j + \eta_j(\boldsymbol{X}_j : \mathbf{m}_j)\,\mathbf{m}_j\}\,\lambda \qquad (j=1,2). \tag{3.17}$$

The material parameters a_j, d_j, η_j have to fulfil

$$a_j > 0, \qquad d_j > 0, \qquad 0 \le \eta_j \le 1 \qquad (j=1,2). \tag{3.18}$$

($d_j = 0$ corresponds to a simpler case.) The tensors $\mathbf{m}_j$ are defined as

$$\mathbf{m}_j := \mathbf{n}_j\,\|\mathbf{n}_j\|^{-1} = \frac{\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*}{\|\boldsymbol{\sigma}_j^* - \boldsymbol{X}_j^*\|} \qquad (j=1,2). \tag{3.19}$$

The isothermal case of this model 2M1C-a (with $a_1 = a_2 = 1$ and $\eta_1 = \eta_2 = 0$) was proposed by Cailletaud and Saï (1995). In Taleb et al. (2006), ratcheting experiments were simulated based on this model. The idea of the projection of $\boldsymbol{X}_j$ onto $\mathbf{m}_j$ is due to Burlet and Cailletaud (1987).

Using the evolution equations (3.7), (3.10) and (3.17) as well as (2.34), (2.37), one can re-write the dissipation inequality (2.29) in the form

$$\begin{aligned}(R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \frac{3d_1}{2c_{11}}(1-\eta_1)\lambda\,\boldsymbol{X}_1 : \boldsymbol{X}_1 + \\ + \frac{3d_1}{2c_{11}}\eta_1\lambda\,(\boldsymbol{X}_1 : \mathbf{m}_1)^2 + \frac{3d_2}{2c_{22}}(1-\eta_2)\lambda\,\boldsymbol{X}_2 : \boldsymbol{X}_2 + \frac{3d_2}{2c_{22}}\eta_2\lambda\,(\boldsymbol{X}_2 : \mathbf{m}_2)^2 \ge 0.\end{aligned} \tag{3.20}$$

Clearly, the model 2M1C-a (characterised by (3.1), (3.7), (3.10), (3.17)) is thermodynamically consistent, if the dissipation inequality (3.20) holds. In Wolff and Taleb (2008), the special case $r = 1$ has been considered. The following theorem covers the more general case.

Theorem 3.3. Assume (3.2) - (3.4), (3.11), (3.18).
(i) In the case of

$$a_1 = a_2 = 1, \tag{3.21}$$

the model 2M1C-a is thermodynamically consistent, if

$$r \le 1 + 2\sqrt{\frac{b\,R_0}{Q}} \tag{3.22}$$

holds.
(ii) In the general case

$$a_1 \ne 1, \qquad a_2 \ne 1 \tag{3.23}$$

the model 2M1C-a is thermodynamically consistent, if

$$\eta_1 < 1, \quad \eta_2 < 1, \tag{3.24}$$

$$\frac{c_{11}}{d_1(1-\eta_1)}\,|1-a_1|^2 + \frac{c_{22}}{d_2(1-\eta_2)}\,|1-a_2|^2 \le 4R_0, \tag{3.25}$$

$$r \le 1 + \sqrt{\frac{b}{Q}\left(4R_0 - \frac{c_{11}}{d_1(1-\eta_1)}\,|1-a_1|^2 - \frac{c_{22}}{d_2(1-\eta_2)}\,|1-a_2|^2\right)} \tag{3.26}$$

Before proving Theorem 3.3, we provide some preliminary results.

Lemma 3.4. (i) Let be r, b, Q, $R_0 > 0$. Then there holds the equivalence

$$\Big(\forall\,R \ge 0 \quad : \quad R_0 + (1-r)R + \frac{b}{Q}R^2 \ge 0\Big) \quad \Leftrightarrow \quad r \le 1 + 2\sqrt{\frac{R_0 b}{Q}} \tag{3.27}$$

(ii) (Young's inequality with a small factor)

$$\forall\,a, b \in \mathbb{R} \quad \forall\,\delta > 0 \quad : \quad |ab| \le \frac{\delta}{2}\,a^2 + \frac{1}{2\,\delta}\,b^2 \tag{3.28}$$

Proof. (of Theorem 3.3) The strategy is to estimate the left-hand side of (3.20) from below by simpler expressions, and to show, that, at the end, the last expression is non-negative.

At first, we note that terms containing $(\boldsymbol{X}_j : \mathbf{m}_j)^2$ are non-negative. Hence, they can be omitted in (3.20). Therefore, it is sufficient to prove the validity of

$$
\begin{aligned}
(3.29)\quad & (R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \\
& + \frac{3d_1}{2c_{11}}(1-\eta_1)\,\lambda \boldsymbol{X}_1 : \boldsymbol{X}_1 + \frac{3d_2}{2c_{22}}(1-\eta_2)\,\lambda \boldsymbol{X}_2 : \boldsymbol{X}_2 \geq 0.
\end{aligned}
$$

Clearly, in the simple case $a_1 = a_2 = 1$, (3.29) is valid, if

$$
(3.30)\qquad R_0 + (1-r)R + \frac{b}{Q}R^2 \geq 0 \qquad\qquad \forall R \geq 0.
$$

Due to (3.27), this is the case, because of the assumption (3.22).

In the general case, the terms containing $\boldsymbol{X}_j : \dot{\boldsymbol{\varepsilon}}_j$ are not definite. But, there is a hope to compensate their behavior by the definiteness of the remaining terms. Using (2.32), (2.37), (3.7) as well as Young's inequality (3.28) and Cauchy-Schwarz inequality, one gets the following estimates:

$$
\begin{aligned}
& |(1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1| = |(1-a_1)\boldsymbol{X}_1 : (\lambda \mathbf{n}_1)| = \\
& = \frac{3}{2}|1-a_1| \left\{ \sqrt{\lambda}\frac{J_1^{N-2}}{J^{N-1}}\|\boldsymbol{\sigma}_1^* - \boldsymbol{X}_1^*\| \right\} : \left\{ \sqrt{\lambda}\|\boldsymbol{X}_1\| \right\} \leq \\
(3.31)\quad & \leq |1-a_1|\lambda \frac{\delta_1}{2}\left(\frac{J_1}{J}\right)^{2(N-1)} + \frac{3|1-a_1|}{4\delta_1}\lambda\|\boldsymbol{X}_1\|^2 \leq \\
& \leq |1-a_1|\lambda\frac{\delta_1}{2} + \frac{3|1-a_1|}{4\delta_1}\lambda\|\boldsymbol{X}_1\|^2,
\end{aligned}
$$

where $\delta_1 > 0$ will be appropriately chosen in 3.29. Analogously, one obtains

$$
(3.32)\qquad |(1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2| = |(1-a_2)\boldsymbol{X}_2 : (\lambda\,\mathbf{n}_2)| \leq |1-a_2|\lambda\frac{\delta_2}{2} + \frac{3|1-a_2|}{4\delta_2}\lambda\|\boldsymbol{X}_2\|^2
$$

for some $\delta_2 > 0$ (cf. 3.31). From (3.29), (3.31), (3.32), one gets

$$
\begin{aligned}
(3.33)\quad & (R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda + (1-a_1)\boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + (1-a_2)\boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \\
& + \frac{3d_1}{2c_{11}}(1-\eta_1)\lambda \boldsymbol{X}_1 : \boldsymbol{X}_1 + \frac{3d_2}{2c_{22}}(1-\eta_2)\lambda \boldsymbol{X}_2 : \boldsymbol{X}_2 + \\
& \geq (R_0 - |1-a_1|\lambda\frac{\delta_1}{2} - |1-a_2|\lambda\frac{\delta_2}{2} + (1-r)R + \frac{b}{Q}R^2)\lambda + \\
& + \frac{3d_1}{2c_{11}}(1-\eta_1)\lambda\|\boldsymbol{X}_1\|^2 + \frac{3d_2}{2c_{22}}(1-\eta_2)\lambda\|\boldsymbol{X}\|^2 - \frac{3|1-a_1|}{4\delta_1}\lambda\|\boldsymbol{X}_1\|^2 - \frac{3|1-a_2|}{4\delta_2}\lambda\|\boldsymbol{X}_2\|^2.
\end{aligned}
$$

As R, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent of each other, it is reasonable to require assumption (3.24). Now, we chose δ_1 and δ_2 such, that the last four terms cancel each other. This can be done by setting

$$
(3.34)\qquad \delta_1 := \frac{|1-a_1|c_{11}}{2(1-\eta_1)d_1}, \qquad\qquad \delta_2 := \frac{|1-a_2|c_{22}}{2(1-\eta_2)d_2}.
$$

This implies from (3.33): (3.33) can be read as

$$
(3.35)\qquad (R_0 - \frac{|1-a_1|^2 c_{11}}{4(1-\eta_1)d_1} - \frac{|1-a_2|^2 c_{22}|}{4(1-\eta_2)d_2} + (1-r)R + \frac{b}{Q}R^2)\lambda \geq 0.
$$

Clearly, it is necessary, that

$$
(3.36)\qquad R^* := R_0 - \frac{|1-a_1|^2 c_{11}}{4(1-\eta_1)d_1} - \frac{|1-a_2|^2 c_{22}}{4(1-\eta_2)d_2} \geq 0.
$$

This is assumption (3.25)! It remains to ensure that

$$R^* + (1-r)R + \frac{b}{Q}R^2 \geq 0 \qquad \text{for all } R \geq 0. \tag{3.37}$$

Obviously, (3.26) is sufficient for (3.37). □

Therefore, in the "trivial case" $a_1 = a_2 = 1$, $r \leq 1$, the model 2M1C-a is thermodynamically consistent. Generally, Theorem 3.3 ensures thermodynamic consistency, if $\eta_j < 1$, and, if the a_j do not differ "too much" from 1, and, if r is not "too much greater" than 1.

Remark 3.5. Theorem 3.3 is also valid in the viscoplastic case (cf. Remark 3.2). The viscoplastic multiplier is only positive, if $J > R_0 + R$, while the plastic multiplier is only positive, if $J = R_0 + R$. Hence, the validity of (3.20) is also sufficient for thermodynamic consistency in the viscoplastic case.

3.3 The model 2M1C-b

Again, we assume (3.1) - (3.4) and (3.7) - (3.11). Contrary to the 2M1C-a model in Subsection 3.2, instead of (3.17), the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are given by

$$\dot{\boldsymbol{\alpha}}_j = a_j\,\dot{\boldsymbol{\varepsilon}}_j - \{(1-\eta_j)\,\boldsymbol{\alpha}_j + \eta_j(\boldsymbol{\alpha}_j : \mathbf{m}_j)\,\mathbf{m}_j\}\,d_j\,\lambda \qquad (j = 1, 2). \tag{3.38}$$

That means, in the right-hand side of (3.38), the backstresses $\boldsymbol{X}_j$ are substituted by the internal variables $\boldsymbol{\alpha}_j$. This approach was proposed in Taleb et al. (2006) in order to get a better description of ratcheting behavior. Analogously, we let the parameters a_j, d_j and η_j fulfil the conditions (3.18). The m_j are defined by (3.19).

Remark 3.6. Contrary to the corresponding evolution equations for the 2MnC-a models (as well as the 1M models), in (3.38) and (4.22), the factor 3/2 does not appear. This way, we keep the notation in Taleb et al. (2006), Wolff and Taleb (2008) and Hassan et al. (2008).

Using the evolution equations (3.7), (3.10) and (3.38) as well as (2.34), (2.37) and (3.5), one can re-write the dissipation inequality (2.29) in the form

$$\begin{aligned}(R_0 + (1-r)R + \frac{b}{Q}R^2)\lambda &+ \frac{2}{3}d_1\lambda(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : \{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1\} + \\ &+ \frac{2}{3}(1-a_1)(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : (\lambda\mathbf{n}_1) + \frac{2}{3}(1-a_2)(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : (\lambda\mathbf{n}_2) + \\ &+ \frac{2}{3}d_2\lambda(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : \{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2\} \geq 0.\end{aligned} \tag{3.39}$$

The case $a_1 = a_2 = 1$, $r = 1$ and $\eta_1 = \eta_2$ is dealt with in Wolff and Taleb (2008). In the general case, there arise more complicated conditions to ensure thermodynamic consistency.

Theorem 3.7. Let be given the assumptions (3.2) - (3.4), (3.11), (3.18). The model 2M1C-b is thermodynamically consistent, if

$$r \leq 1 \tag{3.40}$$

$$\eta_1 < 1, \quad \eta_2 < 1, \tag{3.41}$$

$$c_{11}^2(1-a_1)^2 + c_{12}^2(1-a_2)^2 < R_0 d_1 c_{11}(1-\eta_1), \tag{3.42}$$

$$c_{12}^2(1-a_1)^2 + c_{22}^2(1-a_2)^2 < R_0 d_2 c_{22}(1-\eta_2), \tag{3.43}$$

$$\begin{aligned}c_{12}^2(d_1+d_2)^2 \leq 4\,(d_1c_{11}(1-\eta_1) &- \frac{1}{R_0}(c_{11}^2(1-a_1)^2 + c_{12}^2(1-a_2)^2))\,\cdot \\ &\cdot\,(d_2c_{22}(1-\eta_2) - \frac{1}{R_0}(c_{12}^2(1-a_1)^2 + c_{22}^2(1-a_2)^2)).\end{aligned} \tag{3.44}$$

The proof of Theorem 3.7 is similar to the proof of Theorem 3.3, but more complex. Additionally, one needs a result about quadratic forms.

Lemma 3.8. Let φ be a quadratic form on $\mathbb{R}^{n^2}$ defined by

$$\varphi(\alpha_1, \alpha_2) := c_{11}\, \alpha_1 : \alpha_1 + 2c_{12}\, \alpha_1 : \alpha_2 + c_{22}\, \alpha_2 : \alpha_2 = \\ = c_{11} \sum_{i,j=1}^{m} \alpha_{1ij} : \alpha_{1ij} + 2c_{12} \sum_{i,j=1}^{m} \alpha_{1ij} : \alpha_{2ij} + c_{22} \sum_{i,j=1}^{m} \alpha_{2ij} : \alpha_{2ij} \quad \forall\, \alpha_1, \alpha_2 \in \mathbb{R}^{n^2}, \tag{3.45}$$

where $c_{11}, c_{12}, c_{22} \in \mathbb{R}$. Then the following two assertions are equivalent
(i) $\forall\, \alpha_1, \alpha_2 \in \mathbb{R}^{n^2} \quad : \quad \varphi(\alpha_1, \alpha_2) \geq 0$
(ii) $c_{11} \geq 0, \qquad c_{12}^2 \leq c_{11} c22$

The proof of Lemma 3.8 is elementary, we refer to the Appendix of Wolff and Taleb (2008), e.g.

Remark 3.9. In the simpler case $a_1 = a_2 = 1$ and $r = 1$ (cf. Wolff and Taleb (2008)), the above 2M1C-b model is thermodynamically consistent, if (3.41) holds and if

$$(d_1 - d_2)^2 \leq 4\, d_1\, d_2\, \frac{c_{11} c_{22} (1 - \eta_1)(1 - \eta_2) - c_{12}^2}{c_{12}^2}. \tag{3.46}$$

That means, contrary to the 2M1C-a model, the condition (3.46) restricts η_1 and η_2 even in the simpler case $a_1 = a_2 = 1$.

Remark 3.10. In the case $r > 1$, more complex conditions are sufficient for thermodynamic consistency which involve b and Q.

4 Description and thermodynamic consistency of some 2M2C models

In an analogous way, we investigate two types of 2M2C models. Again, we collect the common features in one subsection.

4.1 Common features of 2M2C models

We assume for the inelastic part ψ_{in} of the free energy (cf. (2.38) and (3.1)):

$$\psi_{in}(\alpha_1, \alpha_2, q_1, q_2, \theta) := \frac{1}{3\varrho} \{c_{11}(\theta)\, \boldsymbol{\alpha_1} : \boldsymbol{\alpha}_1 + 2\, c_{12}(\theta)\, \boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + c_{22}(\theta)\, \boldsymbol{\alpha}_2 : \boldsymbol{\alpha_2}\} + \\ + \frac{1}{2\varrho} \left\{Q_{11}(\theta)\, q_1^2 + 2\, Q_{12}(\theta)\, q_1 q_2 + Q_{22}(\theta)\, q_2^2\right\}. \tag{4.1}$$

Now, there are two scalar internal variables q_1 and q_2 of strain type. They correspond to isotropic hardening in each mechanism. The coefficient Q_{12} stands for a possible interaction of these two kinds of isotropic hardening (cf. Cailletaud and Saï (1995)). Possible interactions of isotropic and kinematic hardening within ψ_{in} will be considered in Section 6.

Remark 4.1. The inelastic free energy ψ_{in} in (3.1) is convex (for frozen temperature), if for all admissible θ

$$c_{11}(\theta) \geq 0, \qquad c_{12}^2(\theta) \leq c_{11}(\theta)\, c_{22}(\theta), \tag{4.2}$$

$$Q_{11}(\theta) \geq 0, \qquad Q_{12}^2(\theta) \leq Q_{11}(\theta)\, Q_{22}(\theta), \tag{4.3}$$

Again, in order to avoid simple cases, we restrict ourselves to

$$c_{11} > 0, \qquad c_{22} > 0, \qquad Q_{11} > 0, \qquad Q_{22} > 0. \tag{4.4}$$

In accordance with (2.27), (2.28) and (4.1), the backstresses $\mathbf{X}_1$ and $\mathbf{X}_2$ as well as R_1 and R_2 fulfill

$$\boldsymbol{X}_1 = \frac{2}{3}\, c_{11}\, \boldsymbol{\alpha}_1 + \frac{2}{3}\, c_{12}\, \boldsymbol{\alpha}_2, \qquad \boldsymbol{X}_2 = \frac{2}{3}\, c_{12}\, \boldsymbol{\alpha}_1 + \frac{2}{3}\, c_{22}\, \boldsymbol{\alpha}_2, \tag{4.5}$$

$$R_1 = Q_{11}\, q_1 + Q_{12}\, q_2, \qquad R_2 = Q_{12}\, q_1 + Q_{22}\, q_2. \tag{4.6}$$

We assume the subsequent evolution equations for the inelastic strains:

$$\dot{\boldsymbol{\varepsilon}}_j = \lambda_j \, \mathbf{n}_j \qquad (\mathbf{n}_j \text{ defined by (2.45) } j = 1, 2), \tag{4.7}$$

The plastic multipliers for both mechanisms $\lambda_j \geq 0$ have to fulfil

$$\lambda_j = 0, \qquad \text{if} \qquad f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_0 j) < 0, \tag{4.8}$$

$$\lambda_j \geq 0, \qquad \text{if} \qquad f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_0 j) = 0 \qquad \text{(flow conditions)}. \tag{4.9}$$

Moreover, we assume evolution laws for q_j:

$$\dot{q}_j = r_j \, \lambda_j - \frac{b_j}{Q_{jj}} R_j \, \lambda_j \qquad (j = 1, 2). \tag{4.10}$$

The material parameters b_j, r_j are assumed to fulfil

$$r_j > 0, \qquad b_j > 0, \qquad (j = 1, 2). \tag{4.11}$$

(Again, we neglect the simpler case $b_j = 0$.) (2.23), (2.31), (2.45) and (4.7) yield

$$\lambda_j = \dot{s}_j \qquad (j = 1, 2). \tag{4.12}$$

4.2 The model 2M2C-a

Now, we deal with some specifics of the 2M2C-a model. Analogously to Subsection 3.2, we assume the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$:

$$\dot{\boldsymbol{\alpha}}_j = a_j \dot{\boldsymbol{\varepsilon}}_j - \frac{3d_j}{2c_{jj}} \left\{ (1 - \eta_j) \boldsymbol{X}_j + \eta_j (\boldsymbol{X}_j : \mathbf{m}_j) \mathbf{m}_j \right\} \lambda_j \qquad (j = 1, 2). \tag{4.13}$$

The m_j are defined by (3.19), and the material parameters a_j, d_j, η_j must fulfil (cf. (3.18))

$$a_j > 0, \qquad d_j > 0, \qquad 0 \leq \eta_j \leq 1 \qquad (j = 1, 2). \tag{4.14}$$

($d_j = 0$ corresponds to a simpler case, again.)

Repeating arguments as above, the dissipation inequality is

$$\begin{aligned}(R_{01} + (1 - r_1) R_1 + \frac{b_1}{Q_{11}} R_1^2) \lambda_1 + (R_{02} + (1 - r_2) R_2 + \frac{b_2}{Q_{22}} R_2^2) \lambda_2 + (1 - a_1) \boldsymbol{X}_1 : \dot{\boldsymbol{\varepsilon}}_1 + \\ + (1 - a_2) \boldsymbol{X}_2 : \dot{\boldsymbol{\varepsilon}}_2 + \frac{3d_1}{2c_{11}} \left\{ (1 - \eta_1) \boldsymbol{X}_1 : \boldsymbol{X}_1 + \eta_1 (\boldsymbol{X}_1 : \mathbf{m}_1)^2 \right\} \lambda_1 + \\ + \frac{3d_2}{2c_{22}} \left\{ (1 - \eta_2) \boldsymbol{X}_2 : \boldsymbol{X}_2 + \eta_2 (\boldsymbol{X}_2 : \mathbf{m}_2)^2 \right\} \lambda_2 \geq 0.\end{aligned} \tag{4.15}$$

Thermodynamic consistency can be ensured similarly as in the case of the 2M1C-a model. Since there are *two* multipliers ($\lambda_j = \dot{s}_j$, $j = 1{,}2$), there is some "decoupling" (cf. Theorem 3.3).

Theorem 4.2. Assume (4.2) - (4.4), (4.11) and (4.14).
(i) In the case

$$a_1 = a_2 = 1, \tag{4.16}$$

the model 2M2C-a is thermodynamic consistent, if

$$r_j \leq 1 + 2 \sqrt{\frac{b_j R_{0j}}{Q_{jj}}} \qquad (j = 1, 2) \tag{4.17}$$

(ii) In the general case

$$a_j \neq 1 \qquad \text{for one or both } j, \tag{4.18}$$

the model 2M1C-a is thermodynamic consistent, if

$$\eta_j < 1 \qquad \text{for the same } j \text{ as in (4.18)}, \tag{4.19}$$

$$\frac{c_{jj}}{d_j (1 - \eta_j)} |1 - a_j|^2 \leq 4 R_{0j} \qquad \text{for the same } j \text{ as in (4.18)}, \tag{4.20}$$

$$r_j \leq 1 + \sqrt{\frac{b_j}{Q_{jj}} \left(4 R_{0j} - \frac{c_{jj}}{d_j (1 - \eta_j)} |1 - a_j|^2 \right)} \qquad \text{for the same } j \text{ as in (4.18)}. \tag{4.21}$$

As for the 2M1C-a model, there is a "trivial case" for the 2M2C-a model: $a_j = 1$, $r_j \leq 1$ (cf. Theorem 3.3). Generally, Theorem 4.2 ensures thermodynamic consistency, if $\eta_j < 1$, and, if a_j do not differ "too much" from 1, and, if r_j is not "too much greater" than 1. Contrary to Theorem 3.3 for the 2M1C-a model, in Theorem 4.2, the conditions for $j = 1$ and $j = 2$ are separated (cf. (4.18)-(4.21)).

4.3 The model 2M2C-b

Now, we investigate the formal two-criteria analogue to the 2M1C-b model, *letting* (4.7) *and* (4.10) *be the same*. Additionally, in (4.13), one could substitute $\boldsymbol{X}_j$ by $\boldsymbol{\alpha}_j$, analogously as in the case of 1C models. Unfortunately, then it becomes very difficult to prove thermodynamic consistency. Hence, instead of (4.13), we assume the following evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$

$$\dot{\boldsymbol{\alpha}}_1 = a_1\, \dot{\boldsymbol{\varepsilon}}_1 - \{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1 + d_{12}\,\boldsymbol{\alpha}_2\}\, d_1\, \lambda_1, \tag{4.22}$$

$$\dot{\boldsymbol{\alpha}}_2 = a_2\, \dot{\boldsymbol{\varepsilon}}_2 - \{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2 + d_{21}\,\boldsymbol{\alpha}_1\}\, d_2\, \lambda_2. \tag{4.23}$$

a_j, d_j and η_j are supposed to satisfy (4.14); see (3.19) for $\mathbf{m}_j$. For the new material parameters d_{12} and d_{21} we assume

$$d_{12} \neq 0, \quad d_{21} \neq 0. \tag{4.24}$$

Using arguments as above, we obtain from (2.40) the dissipation inequality in the specific form of our 2M1C-b model:

$$\begin{aligned}(R_{01} + (1-r_1)R_1 + \frac{b_1}{Q_{11}}R_1^2)\lambda_1 + (R_{02} + (1-r_2)R_2 + \frac{b_2}{Q_{22}}R_2^2)\lambda_2 + \\ + \frac{2}{3}(1-a_1)(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : (\lambda_1\mathbf{n}_1) + \frac{2}{3}(1-a_2)(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : (\lambda_2\mathbf{n}_2) + \\ + \frac{2}{3}d_1\lambda_1(c_{11}\boldsymbol{\alpha}_1 + c_{12}\boldsymbol{\alpha}_2) : \{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1 + d_{12}\boldsymbol{\alpha}_2\} + \\ + \frac{2}{3}d_2\lambda_2(c_{12}\boldsymbol{\alpha}_1 + c_{22}\boldsymbol{\alpha}_2) : \{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2 + d_{21}\boldsymbol{\alpha}_1\} \geq 0.\end{aligned} \tag{4.25}$$

Remark 4.3. (i) Generally, for 2C models one has $\lambda_1 \neq \lambda_2$. Therefore, if $d_{12} = d_{21} = 0$, some (for the mathematical argument needed) quadratic terms cease to exist in (4.25). Hence, in comparison with (3.39) (and with the exception $c_{12} = 0$), it is more difficult to fulfil the inequality (4.25).
(ii) The coupling in the evolution equations (4.22), (4.23) is a *new item* in the modelling of 2M models and indicates possible further generalizations.

Theorem 4.4. Assume (4.2) - (4.4), (4.11) and (4.24). The model 2M2C-b is thermodynamically consistent, if

$$r_1 \leq 1, \qquad r_2 < 1, \tag{4.26}$$

$$\eta_1 < 1, \qquad \eta_2 < 1, \tag{4.27}$$

$$c_{11}^2(1-a_1)^2 < 2\,R_{01}\,d_1\,c_{11}(1-\eta_1), \qquad c_{12}^2(1-a_1)^2 < 2\,R_{01}\,d_1\,c_{12}\,d_{12}, \tag{4.28}$$

$$c_{22}^2(1-a_2)^2 < 2\,R_{02}\,d_2\,c_{22}(1-\eta_2), \qquad c_{12}^2(1-a_2)^2 < 2\,R_{02}\,d_2\,c_{12}\,d_{21}, \tag{4.29}$$

$$\begin{aligned} d_1^2(|c_{12}| + c_{11}|d_{12}|)^2 \leq \\ \leq 4\left(d_1c_{11}(1-\eta_1) - \frac{1}{2R_{01}}c_{11}^2(1-a_1)^2\right)\left(d_1c_{12}d_{12} - \frac{1}{2R_{01}}c_{12}^2(1-a_1)^2\right),\end{aligned} \tag{4.30}$$

$$\begin{aligned} d_2^2(|c_{12}| + c_{22}|d_{21}|)^2 \leq \\ \leq 4\left(d_2c_{22}(1-\eta_2) - \frac{1}{2R_{02}}c_{22}^2(1-a_2)^2\right)\left(d_1c_{12}d_{21} - \frac{1}{2R_{02}}c_{12}^2(1-a_2)^2\right).\end{aligned} \tag{4.31}$$

Similarly as for the 2M1C-b model, even in the simple case $a_1 = a_2 = 1$, $r_1 \leq 1$, $r_2 \leq 1$, Theorem 4.4 only ensures thermodynamic consistency in the case $\eta_1 < 1$, $\eta_2 < 1$. Besides, (4.30), (4.31) describe smallness conditions with respect to the parameters c_{12}, d_{12}, d_{21} which express the coupling of the two mechanisms.

5 Important relations for the backstresses

It is possible to obtain relations for the isotropic hardenings as well as for the backstresses generalizing the classical Armstrong-Frederick relation. These relations are useful for further mathematical investigations and for numerical simulations. In some cases, the variables q or q_1 and q_2 as well as $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ can be excluded, and *differential* equations can be obtained, even in the case of temperature-dependent parameters. This is very helpful for simulations, when one has to update inelastic quantities in each time step. At first, we consider the isotropic hardening. After this, relations for kinematic hardening are derived.

5.1 Relations concerning isotropic hardening

Because, there is an essential difference between 1C and 2C models, we deal separately with them.

5.1.1 Isotropic hardening in the case of 2M1C models

(3.6) and (3.10) imply an integral equation for R

$$R(t) = Q(t) \left\{ \int_0^t r(\tau)\, \lambda(\tau)\, \mathrm{d}\tau - \int_0^t \frac{b(\tau)}{Q(\tau)} R(\tau)\, \lambda(\tau)\, \mathrm{d}\tau \right\}, \tag{5.1}$$

as well as an ordinary differential equation (ODE) (differentiate the relation (3.6) and express q via the same relation)

$$\dot{R}(t) = Q(t)\, r(t)\, \lambda(t) - \left\{ b(t)\, \lambda(t) - \frac{\dot{Q}(t)}{Q(t)} \right\} R. \tag{5.2}$$

For the sake of notational simplicity, we write $Q(t)$ instead of $Q(\theta(t))$ etc. Besides this, the space variable x is suppressed. The unique solution of (5.2) (for the initial value $R(0) = 0$) is given by

$$R(t) = Q(t) \int_0^t r(s)\, \lambda(s) \exp\left(- \int_0^t b(\tau)\, \lambda(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s. \tag{5.3}$$

Moreover, R is non-negative for $t \geq 0$ (cf. (3.4), (3.11), (3.13)). From (5.3) one obtains the estimate

$$0 < R(t) \leq Q(t) \max\{r\} (\min\{b\})^{-1} (1 - \exp(-\min\{b\} \Lambda(t))) \qquad \text{for } t > 0, \tag{5.4}$$

Λ is the primitive of λ (see (3.14)). Maximum and minimum refer to all admissible temperatures (and possibly other quantities). Clearly, if plastic deformation occurs, $R(t)$ is positive. For constant Q, r and b we have

$$R(\Lambda) = \frac{Q\, r}{b} (1 - \exp(-b\, \Lambda)). \tag{5.5}$$

That means, R is a function of Λ alone. The curve $R = R(\Lambda)$ has the initial slope $Q\, r$, and its saturation value is $(Q\, r)/b$. Besides this, R is an increasing function of Λ, as one can expect in the case of isotropic hardening.

5.1.2 Isotropic hardening in the case of 2M2C models

Any attempt to eliminate q_j in order to obtain relations for R_j leads to a substantial difference with respect to the case of 1C models: A system of integral equations comes up. Using (2.41) and (4.11), one obtains the following system of integral equations for R_1 and R_2.

$$\begin{aligned} R_1(t) = Q_{11}(t) \int_0^t \left(r_1(\tau)\lambda_1(\tau) - \frac{b_1(\tau)}{Q_{11}(\tau)} R_1(\tau)\lambda_1(\tau) \right) \mathrm{d}\tau \, + \\ + Q_{12}(t) \int_0^t \left(r_2(\tau)\lambda_2(\tau) - \frac{b_2(\tau)}{Q_{22}(\tau)} R_2(\tau)\lambda_2(\tau) \right) \mathrm{d}\tau, \end{aligned} \tag{5.6}$$

$$(5.7)\quad R_2(t) = Q_{12}(t)\int_0^t \left(r_1(\tau)\lambda_1(\tau) - \frac{b_1(\tau)}{Q_{11}(\tau)} R_1(\tau)\lambda_1(\tau) \right) \mathrm{d}\tau + \\ + Q_{22}(t)\int_0^t \left(r_2(\tau)\lambda_2(\tau) - \frac{b_2(\tau)}{Q_{22}(\tau)} R_2(\tau)\lambda_2(\tau) \right) \mathrm{d}\tau.$$

Again, the dependence on the space variable x is suppressed, and $Q_{11}(t)$ stands for $Q_{11}(\theta(t))$. In the subsequent cases, one can obtain from (5.6), (5.7) differential equations:

1) For *constant* Q_{ij}, differentiation in (5.6), (5.7) leads to a coupled system of differential equations:

$$(5.8)\qquad \dot{R}_1(t) = Q_{11} r_1(t)\lambda_1(t) + Q_{12} r_2(t)\lambda_2(t) - b_1(t)R_1(t)\lambda_1(t) - Q_{12}\frac{b_2(t)}{Q_{22}} R_2(t)\lambda_2(t),$$

$$(5.9)\qquad \dot{R}_2(t) = Q_{12} r_1(t)\lambda_1(t) + Q_{22} r_2(t)\lambda_2(t) - Q_{12}\frac{b_1(t)}{Q_{11}} R_1(t)\lambda_1(t) - b_2(t)R_2(t)\lambda_2(t).$$

Note that the two systems (5.6), (5.7) and (5.8), (5.9) are equivalent, if one assumes the usual initial condition $R_1(0) = R_2(0) = 0$. In comparison to the case of 1C models, a simple solution of (5.8), (5.9) like (5.3) does not exist. Thus, there is a mathematical challenge to formulate appropriate conditions such that $R_j + R_{0j} > 0$. Furthermore, due to the interaction in the isotropic hardening (if $Q_{12} < 0$), there can be a softening in one mechanism caused by the hardening in the other one.

2) In the regular case

$$(5.10)\qquad \Delta_Q := Q_{11}Q_{22} - Q_{12}^2 > 0 \qquad \text{for all admissable arguments,}$$

from (5.6), (5.7) one gets

$$(5.11)\qquad \int_0^t \left(r_1(\tau)\lambda_1(\tau) - \frac{b_1(\tau)}{Q_11(\tau)} R_1(\tau)\lambda_1(\tau) \right) \mathrm{d}\tau = \frac{1}{\Delta_Q}(Q_{22}R_1 - Q_{12}R_2),$$

$$(5.12)\qquad \int_0^t \left(r_2(\tau)\lambda_2(\tau) - \frac{b_2(\tau)}{Q_22(\tau)} R_2(\tau)\lambda_2(\tau) \right) \mathrm{d}\tau = \frac{1}{\Delta_Q}(Q_{11}R_2 - Q_{12}R_1).$$

Hence, by differentiating (5.6), (5.7) with respect to t, and using (5.11), (5.12) we obtain

$$(5.13)\quad \dot{R}_1(t) = Q_{11}(t) r_1(t)\lambda_1(t) + Q_{12}(t) r_2(t)\lambda_2(t) - b_1(t)R_1(t)\lambda_1(t) - Q_{12}(t)\frac{b_2(t)}{Q_{22}(t)} R_2(t)\lambda_2(t) + \\ + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{11}}{d\theta}(Q_{22}R_1 - Q_{12}R_2) + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{12}}{d\theta}(Q_{11}R_2 - Q_{12}R_1),$$

$$(5.14)\quad \dot{R}_2(t) = Q_{12}(t) r_1(t)\lambda_1(t) + Q_{22}(t) r_2(t)\lambda_2(t) - Q_{12}(t)\frac{b_1(t)}{Q_{11}(t)} R_1(t)\lambda_1(t) - b_2(t)R_2(t)\lambda_2(t) + \\ + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{12}}{d\theta}(Q_{22}R_1 - Q_{12}R_2) + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{22}}{d\theta}(Q_{11}R_2 - Q_{12}R_1).$$

3) In the singular case

$$(5.15)\qquad \Delta_Q := Q_{11}Q_{22} - Q_{12}^2 = 0 \qquad \text{for all admissible arguments,}$$

(4.4) implies

$$(5.16)\qquad k := \frac{Q_{12}}{Q_{11}} = \frac{Q_{22}}{Q_{12}} \neq 0$$

as well as

$$(5.17)\qquad R_2 = kR_1.$$

k is allowed be negative and temperature-dependent. Thus, one obtains an integral equation for R_1

$$(5.18)\quad R_1(t) = Q_{11}(t)\left\{ \int_0^t r_1(\tau)\lambda_1(\tau)\,\mathrm{d}\tau + k(t)\int_0^t r_2(\tau)\lambda_2(\tau)\,\mathrm{d}\tau \right\} + \\ - Q_{11}(t)\int_0^t \frac{b_1(\tau)}{Q_{11}(\tau)} R_1(\tau)\lambda_1(\tau)\,\mathrm{d}\tau - k(t)Q_{11}(t)\int_0^t \frac{b_2(\tau)}{Q_{22}(\tau)} k(\tau)R_1(\tau)\lambda\,\mathrm{d}\tau.$$

Only for constant k one obtains from (5.18) a differential equation for R_1.

5.2 Generalized Armstrong-Frederick relations for the 2MnC-a model

We distinguish between the models 2MnC-a and 2MnC-b (with $n = 1$ or $n = 2$). Concerning the models 2MnC-a, the only difference is that one has *one common* multiplier λ in the case of 1C models, and *two* multipliers λ_1 and λ_2 otherwise. We formulate the subsequent formulas for the 2M2C-a model. Setting $\lambda = \lambda_1 = \lambda_2$, one obtains the case for the 2M1C-a model. (3.5) and (3.17) imply integral equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_1$:

$$\boldsymbol{X}_1(t) = \frac{2}{3}c_{11}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\right\}\lambda_1\,\mathrm{d}\tau\right\} + \\ + \frac{2}{3}c_{12}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\right\}\lambda_1\,\mathrm{d}\tau\right\}, \tag{5.19}$$

$$\boldsymbol{X}_2(t) = \frac{2}{3}c_{12}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\right\}\lambda_1\,\mathrm{d}\tau\right\} + \\ + \frac{2}{3}c_{22}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\right\}\lambda_1\,\mathrm{d}\tau\right\}. \tag{5.20}$$

Note: 5.19 and 5.20 do *not* involve $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. Analogously as in the case of two isotropic hardenings, R_1 and R_2, in Subsection 5.1.2, one can derive differential equations. This follows from (5.19), (5.20) some under additional conditions:

1) For *constant* c_{11}, c_{12}, c_{22} one can differentiate (5.19), (5.20) with respect to time t. This yields

$$\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 - d_1\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1 + \\ - \frac{c_{12}d_2}{c_{22}}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2, \tag{5.21}$$

$$\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 - \frac{c_{12}d_1}{c_{11}}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1 + \\ - d_2\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2. \tag{5.22}$$

These last two equations generalize the Armstrong-Frederick equation (cf. Armstrong and Frederick (1966), Lemaitre and Chaboche (1990), Haupt (2002) e.g.) as well as the approach by Burlet and Cailletaud (1987). Indeed, in the case of only one inelastic strain (i.e. $\boldsymbol{\varepsilon}_{in} = \boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2 = 0$, $\boldsymbol{\alpha}_2 = 0$, $\boldsymbol{X}_1 = \boldsymbol{X}$, $\boldsymbol{X}_2 = 0$, $\lambda = \dot{s}_{in}$), (5.21) reduces to

$$\dot{\boldsymbol{X}} = \frac{2}{3}\,c\,a\,\dot{\boldsymbol{\varepsilon}}_{in} - d\{(1-\eta)\boldsymbol{X} + \eta(\boldsymbol{X} : \mathbf{m})\mathbf{m}\}\dot{s}_{in}. \tag{5.23}$$

Finally, for $\eta = 0$, (5.23) turns into the classical Armstrong-Frederick relation; for $\eta = 1$, one gets the proposal by Burlet and Cailletaud (1987).

2) In the regular case

$$\Delta_c := c_{11}c_{22} - c_{12}^2 > 0 \qquad \text{for all admissible arguments,} \tag{5.24}$$

the brackets {} in (5.19), (5.20) can be expressed by $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$:

$$\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_1}{2c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\right\}\lambda_1\,\mathrm{d}\tau\right\} = \frac{3}{2\Delta_c}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2), \tag{5.25}$$

$$\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{3d_2}{2c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\right\}\lambda_2\,\mathrm{d}\tau\right\} = \frac{3}{2\Delta_c}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1). \tag{5.26}$$

Now, analogously to Subsection 5.1.2, one gets differential equations not containing $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$:

$$\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 - d_1\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1 + \\ - \frac{c_{12}}{d_2}c_{22}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2 + \\ + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{11}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1), \tag{5.27}$$

$$\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 - \frac{c_{12}}{d_1}c_{11}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1 + \\ - d_2\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2 + \\ + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{22}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1). \tag{5.28}$$

3) Finally, the singular case

$$\Delta_c := c_{11}c_{22} - c_{12}^2 = 0 \qquad \text{for all admissible arguments} \tag{5.29}$$

can be dealt with analogously as the singular case in Subsection 5.1.2.

5.3 Generalized Armstrong-Frederick relations for the 2MnC-b model

Since the 2M2C-b model is more complex than the 2M1C-b model (cf. (4.22), (4.23)), we write down only the expressions for the 2M1C-b model. Analogously to Subsection 5.2, from (3.5) and (3.38) we obtain integral equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$:

$$\boldsymbol{X}_1(t) = \frac{2}{3}c_{11}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_1}{c_{11}}\{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda\,\mathrm{d}\tau\right\} + \\ + \frac{2}{3}c_{12}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_2}{c_{22}}\{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda\,\mathrm{d}\tau\right\}, \tag{5.30}$$

$$\boldsymbol{X}_2(t) = \frac{2}{3}c_{12}(t)\left\{\int_0^t a_1(\tau)\dot{\boldsymbol{\varepsilon}}_1(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_1}{c_{11}}\{(1-\eta_1)\boldsymbol{\alpha}_1 + \eta_1(\boldsymbol{\alpha}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda\,\mathrm{d}\tau\right\} + \\ + \frac{2}{3}c_{22}(t)\left\{\int_0^t a_2(\tau)\dot{\boldsymbol{\varepsilon}}_2(\tau)\,\mathrm{d}\tau - \int_0^t \frac{d_2}{c_{22}}\{(1-\eta_2)\boldsymbol{\alpha}_2 + \eta_2(\boldsymbol{\alpha}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda\,\mathrm{d}\tau\right\}. \tag{5.31}$$

An elimination of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ is only possible under the additional condition (5.24). Then the equations in (3.5) are uniquely solvable with respect to $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$:

$$\boldsymbol{\alpha}_1 = \frac{3}{2\Delta_c}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2), \qquad \boldsymbol{\alpha}_2 = \frac{3}{2\Delta_c}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1). \tag{5.32}$$

Inserting (5.32) into (5.30), (5.31), one obtains integral equations not containing $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.
Again, for constant c_{ij} one can take the derivatives with respect to t and obtains the following generalizations of Armstrong-Frederick relations:

$$\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 + \\ - c_{11}\frac{d_1}{\Delta_c}\{(1-\eta_1)(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \eta_1(c_{22}(\boldsymbol{X}_1 : \mathbf{m}_1) - c_{12}(\boldsymbol{X}_2 : \mathbf{m}_1))\mathbf{m}_1\}\lambda + \\ - c_{12}\frac{d_2}{\Delta_c}\{(1-\eta_2)(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1) + \eta_2(c_{11}(\boldsymbol{X}_2 : \mathbf{m}_2) - c_{12}(\boldsymbol{X}_1 : \mathbf{m}_2))\mathbf{m}_2\}\lambda, \tag{5.33}$$

$$\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 + \\ - c_{12}\frac{d_1}{\Delta_c}\{(1-\eta_1)(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \eta_1(c_{22}(\boldsymbol{X}_1 : \mathbf{m}_1) - c_{12}(\boldsymbol{X}_2 : \mathbf{m}_1))\mathbf{m}_1\}\lambda + \\ - c_{22}\frac{d_2}{\Delta_c}\{(1-\eta_2)(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1) + \eta_2(c_{11}(\boldsymbol{X}_2 : \mathbf{m}_2) - c_{12}(\boldsymbol{X}_1 : \mathbf{m}_2))\mathbf{m}_2\}\lambda. \tag{5.34}$$

Remark 5.1. **(i)** In the case of constant c_{ij}, the Armstrong-Frederick relations (5.21), (5.22) and (5.33), (5.34) have a similar structure. But, in the case of 2M1C-b model, in (5.33), (5.34), there are the additional coupling terms $(\boldsymbol{X}_2 : \mathbf{m}_1))\mathbf{m}_1, (\boldsymbol{X}_1 : \mathbf{m}_2))\mathbf{m}_2$.
(ii) In the case of the 2M2C-b model, one gets similar integral equations as in (5.30), (5.31). In the regular case (5.24), $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ can be excluded.
(iii) In the regular case (5.24), one can get elaborated differential equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, if some of the c_{ij} depend on the temperature.

5.4 Some mathematical consequences of the Armstrong-Frederick relations

5.4.1 Tracelessness of backstresses and of kinematic variables

Here, the inelastic strains $\boldsymbol{\varepsilon}_{in}$ as well as $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are assumed to be traceless (cf. (2.6), (2.21)). Hence, it is reasonable to ask whether the backstresses $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ as well as $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are traceless, too. One can prove the following result.

Theorem 5.2. Under the assumption (5.24), there hold

$$\mathrm{tr}(\boldsymbol{X}_1) = \mathrm{tr}(\boldsymbol{X}_2) = \mathrm{tr}(\boldsymbol{\alpha}_1) = \mathrm{tr}(\boldsymbol{\alpha}_2) = 0 \tag{5.35}$$

for both 2MnC-a and 2MnC-b models.

Proof. In case of a 2MnC-a model, $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ fulfil the system of integral equations (5.19), (5.20). These are linear Volterra integral equations. The general mathematical theory says that there is a unique solution ($\boldsymbol{X}_1$, $\boldsymbol{X}_2$) for given $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$. (The relation (3.19) for the quantities $\mathbf{m}_1$ and $\mathbf{m}_2$ is taken into account.) As $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\sigma}_1^*$, $\boldsymbol{\sigma}_2^*$ are traceless, the deviators $\boldsymbol{X}_1^*$ and $\boldsymbol{X}_2^*$ also fulfil the same system of Volterra equations (5.19), (5.20). Due to uniqueness, there must be $\boldsymbol{X}_j = \boldsymbol{X}_j^*$ for $j = 1, 2$. As (5.24) holds, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are linear combinations of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ (cf. (5.32)). And, so they are traceless too.

In the case of 2MnC-b models, after expressing $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ in accordance with (5.32), (5.30), (5.31) becomes a linear Volterra integral equations for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ alone. The further reasoning is as above. ☐

Remark 5.3. (i) In case of 2MnC-a models, one has trace-less backstresses, not assuming (5.24).
(ii) The above results make it reasonable, to assume that the symmetric tensorial internal variables $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are trace-less (at least, if the inelastic strains are traceless). Then, due to (2.27), the backstresses $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are traceless not only in the regular cases.

5.4.2 Equations for the total strain and its rate

We notice further interesting relations. Assuming the regular case (5.24), one gets from (3.5) and (4.5), respectively,

$$\frac{A_1}{a_1}\boldsymbol{\alpha}_1 = \frac{3}{2\Delta_c}\frac{A_1}{a_1}(c_{22} - c_{12})\boldsymbol{X}_1, \qquad \frac{A_2}{a_2}\boldsymbol{\alpha}_2 = \frac{3}{2\Delta_c}\frac{A_2}{a_2}(c_{11} - c_{12})\boldsymbol{X}_2. \tag{5.36}$$

Using the evolution equations for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ for the 2MnC-a models (cf. (3.17) and (4.13), respectively), one obtains an equation for the total inelastic strain:

$$\begin{aligned}\boldsymbol{\varepsilon}_{in} = \frac{3}{2\Delta_c}\left\{\frac{A_1}{a_1}(c_{22} - c_{12})\boldsymbol{X}_1 + \frac{A_2}{a_2}(c_{11} - c_{12})\boldsymbol{X}_2\right\} +& \\ + \int_0^t \frac{3A_1 d_1}{2a_1 c_{11}}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1\, \mathrm{d}\tau +& \\ + \int_0^t \frac{3A_2 d_2}{2a_2 c_{22}}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2\, \mathrm{d}\tau.&\end{aligned} \tag{5.37}$$

For 1C models one has to set $\lambda_1 = \lambda_2 = \lambda$. Sometimes, this equation can be useful for estimating the total strain. For 2MnC-b models, an analogous equation can be derived, substituting $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ in (3.38) and (4.22), respectively, via (5.36). Taking (5.24) and (3.4) into account, there hold

$$0 \le |c_{12}| < \sqrt{c_{11}c_{22}} \le \max\{c_{11}, c_{22}\}. \tag{5.38}$$

And therefore, the differences $(c_{22} - c_{12})$ and $(c_{11} - c_{12})$ do not vanish at the same time. Hence, the term outside of the integral on the right-hand side of (5.37) does not vanish, if there is inelastic motion.

In the case of *constant* c_{ij}, a_j, A_j, differentiation of (5.37) gives an equation for the total inelastic strain rate

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}}_{in} = \frac{3}{2\Delta_c}\left\{\frac{A_1}{a_1}(c_{22} - c_{12})\dot{\boldsymbol{X}}_1 + \frac{A_2}{a_2}(c_{11} - c_{12})\dot{\boldsymbol{X}}_2\right\} +& \\ + \frac{3A_1 d_1}{2a_1 c_{11}}\left\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\right\}\lambda_1 +& \\ + \frac{3A_2 d_2}{2a_2 c_{22}}\left\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\right\}\lambda_2&\end{aligned} \tag{5.39}$$

Special cases of (5.38) can be found in Taleb et al. (2006).

In the general case of non-constant c_{ij}, a_j, A_j, one can differentiate (5.37) with respect to t, obtaining generally a quite complex formula. An easier way is to start with differentiation of (3.5), taking (5.32) into account. This gives

$$\dot{\boldsymbol{X}}_1 = \frac{2}{3}c_{11}\dot{\boldsymbol{\alpha}}_1 + \frac{2}{3}c_{12}\dot{\boldsymbol{\alpha}}_2 + \frac{2}{3\Delta_c}\dot{\theta}\frac{dc_{11}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{2}{3\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1), \tag{5.40}$$

$$\dot{\boldsymbol{X}}_2 = \frac{2}{3}c_{12}\dot{\boldsymbol{\alpha}}_1 + \frac{2}{3}c_{22}\dot{\boldsymbol{\alpha}}_2 + \frac{2}{3\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{2}{3\Delta_c}\dot{\theta}\frac{dc_{22}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1). \tag{5.41}$$

Due to (5.24), $\dot{\boldsymbol{\alpha}}_1$ and $\dot{\boldsymbol{\alpha}}_1$ can be excluded. Using the evolution equations for $\boldsymbol{\alpha}_j$ (cf. (3.17), (3.38), (4.13), (4.22)), one gets an equation for the rate of the total inelastic strain.

Remark 5.4. If (5.24) is not fulfilled, one cannot resolve the system (3.5) with respect to $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$. Therefore, in this case, an analogous equation to (5.37) equation contains at least one $\boldsymbol{\alpha}_j$.

6 An extension of 2M models

The 2M models described above are "in use" (besides the new proposal for the 2M2C-b model in (4.22), (4.23)), or, they are simple extensions of such models. Besides, they have been applied to simulation of ratcheting (cf. Saï et al. (2004), Taleb et al. (2006), Hassan et al. (2008) and the references therein) as well as to modelling of material behaviour of steel undergoing phase transformations (cf. Wolff et al. (2008) for details).

Now, we want to present a possible extension concerning the coupling between kinematic and isotropic hardening. Again, we only deal with 2M models, remarking that the subsequent extension can be generally applied to multi-mechanism models.

A general reference to coupling between kinematic and isotropic hardening can be already found in Cailletaud and Saï (1995). Here, we want to give a more concrete example for this. We focus on 2M1C models. Keeping the inelastic part of the free energy as a quadratic form, we propose

$$\psi_{in}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, q, \theta) := \frac{1}{3\varrho}\left\{c_{11}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_1 + 2c_{12}(\theta)\,\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 + c_{22}(\theta)\,\boldsymbol{\alpha}_2 : \boldsymbol{\alpha}_2\right\} + \\ + \frac{1}{2\varrho}\left\{Q(\theta)\,q^2 + 2\,q\,\mathbf{Q}_1(\theta) : \boldsymbol{\alpha}_1 + 2\,q\,\mathbf{Q}_2(\theta) : \boldsymbol{\alpha}_2\right\}. \tag{6.1}$$

Besides the scalar material parameters c_{ij} as above, there arise two matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ playing the role of material parameters. Clearly, for all admissible θ, ψ_{in} must be a convex quadratic form of $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and q with positive diagonal. The Sylvester criterion provides a sufficient condition for that:

$$c_{11} > 0, \qquad c_{22} > 0, \qquad c_{12}^2 \leq c_{11}\,c_{22}, \qquad Q > 0, \tag{6.2}$$

$$Q\,(c_{11}\,c_{22} - c_{12}^2) + \|\mathbf{Q}_2\|\,(|c_{12}|\,\|\mathbf{Q}_1\| - c_{11}\,\|\mathbf{Q}_2\|) + \|\mathbf{Q}_1\|\,(|c_{12}|\,\|\mathbf{Q}_2\| - c_{22}\,\|\mathbf{Q}_1\|) \geq 0 \tag{6.3}$$

for all admissible temperatures. Due to (6.1), there arise the subsequent coupled relations for backstresses and isotropic hardening (cf. (2.27), (2.28)):

$$\boldsymbol{X}_1 = \frac{2}{3}c_{11}\,\boldsymbol{\alpha}_1 + \frac{2}{3}c_{12}\,\boldsymbol{\alpha}_2 + q\,\mathbf{Q}_1, \tag{6.4}$$

$$\boldsymbol{X}_2 = \frac{2}{3}c_{12}\,\boldsymbol{\alpha}_1 + \frac{2}{3}c_{22}\,\boldsymbol{\alpha}_2 + q\,\mathbf{Q}_2, \tag{6.5}$$

$$R = Q\,q + \mathbf{Q}_1 : \boldsymbol{\alpha}_1 + \mathbf{Q}_2 : \boldsymbol{\alpha}_2 \tag{6.6}$$

Now, one can propose evolution equations for $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ and q as in (3.7), (3.10), (3.17). This leads to the dissipation inequality (3.20), and Theorem 3.3 applies analogously.

Alternatively, one can propose several couplings within the evolution equations. For instance, besides (3.7), there can be supposed:

$$\dot{\boldsymbol{\alpha}}_j = a_j\,\dot{\boldsymbol{\varepsilon}}_j - d_j\,\boldsymbol{X}_j\,\lambda - \mathbf{D}_j\,R\,\lambda \qquad (j = 1, 2) \tag{6.7}$$

$$\dot{q} = r\,\lambda - b\,R\,\lambda - \mathbf{B}_1 : \boldsymbol{X}_1\,\lambda - \mathbf{B}_2 : \boldsymbol{X}_2\,\lambda. \tag{6.8}$$

($\mathbf{D}_j$ and $\mathbf{B}_j$ are further (tensorial) material parameters.) Now, in comparison with (3.20), the dissipation inequality has additional terms. Thus, an analogue to Theorem 3.3 requires more conditions.

Moreover, one needs conditions ensuring that $R_0 + R > 0$. Clearly, this leads to further restrictions for $\boldsymbol{Q}_j$ as well as for the parameters in (6.7), (6.8).

Besides, from (6.4) - (6.8) one obtains quite complex Armstrong-Frederick relations, also in the case of constant c_{ij}, Q, $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

7 Mathematical problems for 2M models

We want to formulate mathematical problems resulting from the modelling given above. As usual in structure mechanics, the mathematical challenge consists in determining the fields of displacements and of temperature. Sometimes, there is an interest in finding other quantities like stresses. The modelling of 2M models given above is quite complex. And thus, the resulting mathematical problems keep this complexity. We will sketch this for 2M models under special conditions allowing to exclude the internal variables.

Taking (2.1), (2.4) – (2.6), (2.14), (2.20) into account, one gets the impulse equation in the displacement formulation

$$\varrho\,\ddot{\boldsymbol{u}} - \operatorname{div}(2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mu_L \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} - 3\,K\,\alpha(\theta-\theta_0)\boldsymbol{I}) = \boldsymbol{f} - \operatorname{div}(2\mu\,(A_1\,\boldsymbol{\varepsilon}_1 + A_2\,\boldsymbol{\varepsilon}_2)) \tag{7.1}$$

($\mu_L = K - \frac{2}{3}\mu$ - 2nd Lamé coefficient). We re-write the heat-conduction equation (2.30)

$$\begin{aligned}\varrho\,c_d\,\dot{\theta} - \operatorname{div}(\kappa\,\nabla\theta) =&(\boldsymbol{\sigma}_1 - \boldsymbol{X}_1) : \dot{\boldsymbol{\varepsilon}}_1 + (\boldsymbol{\sigma}_2 - \boldsymbol{X}_2) : \dot{\boldsymbol{\varepsilon}}_2 + \boldsymbol{X}_1 : (\dot{\boldsymbol{\varepsilon}}_1 - \dot{\boldsymbol{\alpha}}_1) + \boldsymbol{X}_2 : (\dot{\boldsymbol{\varepsilon}}_2 - \dot{\boldsymbol{\alpha}}_2) + \\ &- R\,\dot{q} + \theta\,\frac{\partial \boldsymbol{X}_1}{\partial\theta} : \dot{\boldsymbol{\alpha}}_1 + \theta\,\frac{\partial \boldsymbol{X}_2}{\partial\theta} : \dot{\boldsymbol{\alpha}}_2 + \theta\,\frac{\partial R}{\partial\theta}\,\dot{q} + \theta\,\frac{\partial\boldsymbol{\sigma}}{\partial\theta} : \dot{\boldsymbol{\varepsilon}}_{te} + r.\end{aligned} \tag{7.2}$$

(Using (2.14) and (2.25), the stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_j$ can be excluded.)

1) 2M1C models: The evolution of the inelastic strains is given by

$$\dot{\boldsymbol{\varepsilon}}_j = \lambda\,\mathbf{n}_j \tag{7.3}$$

The common plastic multiplier λ has to fulfil

$$\lambda = 0, \quad \text{if} \quad f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) < 0, \tag{7.4}$$

$$\lambda \geq 0, \quad \text{if} \quad f(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{X}_1, \boldsymbol{X}_2, R, R_0) = 0 \qquad \text{(flow condition).} \tag{7.5}$$

As usual, λ can be determined by the flow condition (7.5) and by the consistency condition which follows from (7.5). Usually, in numerical schemes, the plastic multiplier will be simultaneously calculated in each time step.

Considering the "regular" case (5.24), the internal variables $\boldsymbol{\alpha}_j$ $(j = 1, 2)$ can be excluded, and, one obtains the evolution equations for the backstresses (cf. (5.27), (5.28))

$$\begin{aligned}\dot{\boldsymbol{X}}_1 = &\frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 - d_1\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda + \\ &- \frac{c_{12}}{d_2}c_{22}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda + \\ &+ \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{11}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1),\end{aligned} \tag{7.6}$$

$$\begin{aligned}\dot{\boldsymbol{X}}_2 = &\frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 - \frac{c_{12}}{d_1}c_{11}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda + \\ &- d_2\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda + \\ &+ \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{22}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1).\end{aligned} \tag{7.7}$$

Moreover, the evolution for R is given by

$$\dot{R}(t) = Q(t)\,r(t)\,\lambda(t) - \left\{b(t)\,\lambda(t) - \frac{\dot{Q}(t)}{Q(t)}\right\}R. \tag{7.8}$$

2) 2M2C models: The evolution of the inelastic strains is given by

$$\dot{\boldsymbol{\varepsilon}}_j = \lambda_j \,\mathbf{n}_j \quad j = 1,2), \tag{7.9}$$

The plastic multipliers for both mechanisms $\lambda_j \geq 0$ have to fulfil

$$\lambda_j = 0, \qquad \text{if} \qquad f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_0 j) < 0, \tag{7.10}$$

$$\lambda_j \geq 0, \qquad \text{if} \qquad f_j(\boldsymbol{\sigma}_j, \boldsymbol{X}_j, R_j, R_0 j) = 0 \qquad \text{(flow conditions).} \tag{7.11}$$

Again, the multipliers λ_j can be determined by the corresponding flow conditions (7.11) and by the consistency conditions which follows from (7.11).

Considering the "regular" case (5.24), the internal variables $\boldsymbol{\alpha}_j$ $(j = 1,2)$ can be excluded, and, one obtains the evolution equations for the backstresses (cf. (5.27), (5.28))

$$\begin{aligned}\dot{\boldsymbol{X}}_1 = {} & \frac{2}{3}c_{11}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{12}a_2\dot{\boldsymbol{\varepsilon}}_2 - d_1\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1 + \\ & - \frac{c_{12}}{d_2}c_{22}\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2 + \\ & + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{11}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1),\end{aligned} \tag{7.12}$$

$$\begin{aligned}\dot{\boldsymbol{X}}_2 = {} & \frac{2}{3}c_{12}a_1\dot{\boldsymbol{\varepsilon}}_1 + \frac{2}{3}c_{22}a_2\dot{\boldsymbol{\varepsilon}}_2 - \frac{c_{12}}{d_1}c_{11}\{(1-\eta_1)\boldsymbol{X}_1 + \eta_1(\boldsymbol{X}_1 : \mathbf{m}_1)\mathbf{m}_1\}\lambda_1 + \\ & - d_2\{(1-\eta_2)\boldsymbol{X}_2 + \eta_2(\boldsymbol{X}_2 : \mathbf{m}_2)\mathbf{m}_2\}\lambda_2 + \\ & + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{12}}{d\theta}(c_{22}\boldsymbol{X}_1 - c_{12}\boldsymbol{X}_2) + \frac{1}{\Delta_c}\dot{\theta}\frac{dc_{22}}{d\theta}(c_{11}\boldsymbol{X}_2 - c_{12}\boldsymbol{X}_1).\end{aligned} \tag{7.13}$$

Now, there are two isotropic hardenings R_j. In the "regular" case (5.10), there hold (cf. (5.13), (5.14))

$$\begin{aligned}\dot{R}_1(t) = {} & Q_{11}(t)r_1(t)\lambda_1(t) + Q_{12}(t)r_2(t)\lambda_2(t) - b_1(t)R_1(t)\lambda_1(t) - Q_{12}(t)\frac{b_2(t)}{Q_{22}(t)}R_2(t)\lambda_2(t) + \\ & + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{11}}{d\theta}(Q_{22}R_1 - Q_{12}R_2) + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{12}}{d\theta}(Q_{11}R_2 - Q_{12}R_1),\end{aligned} \tag{7.14}$$

$$\begin{aligned}\dot{R}_2(t) = {} & Q_{12}(t)r_1(t)\lambda_1(t) + Q_{22}(t)r_2(t)\lambda_2(t) - Q_{12}(t)\frac{b_1(t)}{Q_{11}(t)}R_1(t)\lambda_1(t) - b_2(t)R_2(t)\lambda_2(t) + \\ & + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{12}}{d\theta}(Q_{22}R_1 - Q_{12}R_2) + \frac{1}{\Delta_Q}\dot{\theta}\frac{dQ_{22}}{d\theta}(Q_{11}R_2 - Q_{12}R_1).\end{aligned} \tag{7.15}$$

The above equations must be fulfilled in a space-time domain $\Omega\times]0,T[$, where Ω is a bounded Lipschitz domain describing the body (workpiece) in its reference configuration, $T > 0$ is the process time.

Finally, one needs initial and boundary conditions. We require, with given u_0, u_1, and θ_0:

$$\boldsymbol{u}(x,0) = u_0(x), \qquad \dot{\boldsymbol{u}}(x,0) = u_1(x), \qquad \theta(x,0) = \theta_0. \tag{7.16}$$

Modelling heat treatment, homogeneous initial conditions for $\boldsymbol{u}$ seem to be reasonable. To obtain correctly posed mathematical problems, there must be boundary conditions for $\boldsymbol{u}$ and θ prescribed, for instance, mixed conditions for $\boldsymbol{u}$:

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_1, \qquad \boldsymbol{\sigma}\,\boldsymbol{\nu} = 0 \quad \text{on } \Gamma_2, \tag{7.17}$$

Γ_1 and Γ_2 are disjoint parts of the whole boundary $\partial\Omega$ of Ω. ν is the outer normal on Γ_2. Concerning the temperature, we put a boundary condition which models the heat exchange:

$$-\kappa\,\frac{\partial\theta}{\partial\boldsymbol{\nu}} = \delta(\theta - \theta_\Gamma) \quad \text{on } \partial\Omega. \tag{7.18}$$

$\delta \geq 0$ is the heat-exchange coefficient, θ_Γ is the ambient temperature.

Moreover, we put initial conditions for $\boldsymbol{\varepsilon}_j$, $\boldsymbol{X}_j$ and R (or R_j):

$$\boldsymbol{\varepsilon}_j(x,0) = 0 \qquad \boldsymbol{X}_j(x,0) = 0, \qquad R_j(x,0) = 0. \tag{7.19}$$

In general, the system of equations and initial and boundary conditions listed above constitutes a quite complex mathematical problem. To our knowledge, existence results are unknown. In Suhr (2010) and Wolff et al. (2010b), a semi-implicit algorithm for numerical simulations for a special 2M model describing the material behaviour of steel has been developed, and simulations have been performed.

8 Conclusion

In this study, we have dealt with two-mechanism models (2M models), focussing on their modelling in the case of plastic behaviour. 2M models with one and with two yield criteria have been considered (see Section 2).

- In Sections 3, 4 some results on thermodynamic consistency have been proved. When considering non-standard cases without dissipation potentials, the proof of thermodynamic consistency is not trivial.
- In Section 5, relations for backstresses and isotropic hardening have been derived generalizing the classical Armstrong-Frederick relations. These relations are very useful for further mathematical investigations including numerical simulations (cf. Wolff et al. (2010b) for a special case).
- In Section 6, an extension concerning coupling between kinematic and isotropic hardening has been given. References to further extensions can be found in Taleb and Cailletaud (2010) and Saï (2010).
- In Section 7, some resulting mathematical problems have been formulated in short. Clearly, the corresponding mathematical investigations remain for future work.

For further discussions and applications of multi-mechanism models we refer to the recent paper by Saï (2010).

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