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Morozov's Discrepancy Principle for Tikhonov regularization of nonlinear operators

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Abstract

We consider Morozov's discrepancy principle for Tikhonov–regularization of nonlinear operator equations. It is shown that minor restrictions to the operator F already guarantee the existence of a regularization parameter α such that $\delta \leq \|y^\delta - F(x_\alpha^\delta)\| \leq c_1\delta$ holds. Moreover, some additional smoothness assumptions on the solution of $F(x) = y$ ensure an optimal convergence rate. Finally we investigate some practically relevant examples, e.g. medical imaging (Single Photon Emission Computed Tomography). It is illustrated that the introduced conditions on F will be met in general by a large class of nonlinear operators.

1 Introduction

A large variety of technical and physical problems can be mathematically modeled by an operator equation

$$F(x) = y \ , \tag{1}$$

where $F: \ X \to Y$ is a (nonlinear) operator between Hilbert spaces, x the searched–for information and y the exact data. Typical examples of such problems arise in medical imaging [22] or inverse scattering [5]. The available data usually stems from a measurement process. Due to measurement errors, we have to deal with noisy data y^δ which satisfy

$$\|y^\delta - y\| \leq \delta \ . \tag{2}$$

If the solution of (1) does not depend continuously on the data, then the problem is called $ill-posed$. In case of inexact data, this instability requires regularization methods for treating the inverse problem.

Because F is a nonlinear operator, equation (1) might have several solutions. We will call x_* an $\bar{x}$-minimum-norm-solution, iff

$$F(x_*) = y \tag{3}$$

and

$$\|x_* - \bar{x}\| = \min_{x \in D(F)} \{\|x - \bar{x}\| : F(x) = y\} \ . \tag{4}$$

During the last decade most of the well known regularization methods for linear operator equations have been generalized to special classes of nonlinear operators. E.g. iterative methods like Landweber iteration [15, 25], Levenberg–Marquardt methods [13], Gauss–Newton [1, 3], conjugate gradient [14] and Newton–like methods [2] are easily to implement. Unfortunately, these methods work only under relatively strong conditions on the nonlinear operator and its Frechét derivative.

Another widely used method is Tikhonov–regularization, where the nonlinear equation (1) is replaced by the minimization problem of finding a minimizer x_α^δ of the Tikhonov functional

$$J_\alpha(x) = \|y^\delta - F(x)\|^2 + \alpha\|x - \bar{x}\|^2 \ . \tag{5}$$

Tikhonov–regularization works for a reasonably large class of nonlinear operators. In principle, it can be applied to weakly sequentially closed operators with Lipschitz–continuous Frechét derivative [10]. As for all regularization methods, a main problem is the choice of the regularization parameter.
To obtain convergence rates for Tikhonov–regularization, one has to assume a smoothness condition $x_* - \bar{x} = F'(x_*)^*\omega$ with sufficiently small $\|\omega\|$. With an a priori parameter choice $\alpha = c\delta$, $c > 0$, a convergence rate

$$\|x_\alpha^\delta - x_*\| \le k(c)\sqrt{\delta} \tag{6}$$

with $k(c) > 0$ can be obtained [11]. An examination of the convergence proof shows that $k(c)$ is minimized by the optimal parameter choice $\alpha = c_{opt}\delta$, $c_{opt} = \|\omega\|^{-1}$, and

$$\|x_\alpha^\delta - x_*\| \le \frac{2\|\omega\|^{1/2}}{(1 - L\|\omega\|)^{1/2}}\sqrt{\delta} \ . \tag{7}$$

(L denotes the Lipschitz–constant for the Frechét derivative). In general, the value of $\|\omega\|$ is not available , and so is c_{opt}. As a consequence, one will never get the optimal constant k for an a priori parameter choice.
An alternative are a posteriori parameter strategies. A well studied method is Morozov's discrepancy principle, where a regularization parameter with

$$\|y^\delta - F(x_\alpha^\delta)\| = c\delta \ , \tag{8}$$

$c \ge 1$, is used. An advantage of Morozov's principle is that, even without knowing $\|\omega\|$, one gets always an estimate

$$\|x_\alpha^\delta - x_*\| \le \frac{(2(1+c)\|\omega\|)^{1/2}}{(1 - L\|\omega\|)^{1/2}}\sqrt{\delta} \tag{9}$$

(see Theorem 2.9). For $c = 1$, we get the optimal error bound (7), for $c > 1$ this bound is multiplied by $\sqrt{(1+c)/2}$. A drawback of the discrepancy principle is that a regularization parameter with (8) might not exist for general nonlinear operators F. Moreover, even if such a parameter exist it requires an additional optimization process to find it numerically. The classical algorithm for Morozov's discrepancy principle applied to Tikhonov regularization for linear operators A [18, 12, 19] would be to choose c_1, $\alpha_0 > 0$, $0 < q < 1$, $\alpha_j = q^j\alpha_0$, and to compute $x_{\alpha_j}^\delta$ until

$$\delta \le \|y^\delta - Ax_{\alpha_j}\| \le c_1\delta \tag{10}$$

holds. Hence, (5) has to be minimized for a set of regularization parameters.
Let us now review some results concerning the discrepancy principle for Tikhonov regularization of nonlinear operators. A parameter choice with (8) and $c = 1$ was considered in [7]. It was shown that the existence of a regularization parameter with (8) is connected with the convexity of a functional in dependence of the noise level and the minimum norm least squares solution of (1) within the given noise level δ. Scherzer [27] considered (8) with $c > 1$. Under the assumption that for every $x,\ z,\ v \in X$ there exists a $k(x, z, v) \in X$ with

$$(F'(x) - F'(z))v \;=\; F'(z)k(x, z, v) \qquad \text{and} \tag{11}$$

$$\|k(x, z, v)\| \;\leq\; K_0\|x - z\|\|v\| \;, \tag{12}$$

it was shown that a parameter α with (8) always exists. In [28] some examples of operators which satisfy these conditions were given, but it seemed to us that it is rather difficult to prove the above conditions for some nonlinear problems of interest, e.g. bilinear operators or operators arising in medical imaging. We might remark that a priori parameter choices will work without these conditions, which means that there is a big gap between the applicability of a priori and a posteriori parameter choices.
The goal of this paper will be a relaxation of (11) and (12). In addition, we will not consider the classical discrepancy principle. For a numerical realization it will not be possible to obtain a parameter α such that (8) holds. As for the linear case, we will choose a trust region $[\delta, c_1\delta]$, $c_1 > 1$, and determine the regularization parameter such that the residual $\|y^\delta - F(x_\alpha^\delta)\|$ belongs to the trust region.
In Section 2 it will be shown that for strongly continuous operators such an regularization parameter always exists, and a convergence rate result is given. Section 3 contains several applications. The first one fails to fulfill conditions (11), (12), but it can be shown easily that our results apply. The last two examples stem from bilinear operator equations and medical imaging (Single Photon Emission Computed Tomography). In both cases it will be demonstrated that our conditions can be met. As for an priori parameter choice, we get these results without severe restrictions to the nonlinear operator.

2 Existence of the regularization parameter

To use Morozov's discrepancy principle, we have to find a parameter α such that

$$\delta \leq \|y^\delta - F(x_\alpha^\delta)\| \leq c_1\delta \qquad c_1 > 1 \tag{13}$$

holds. The problem is that for arbitrary nonlinear operators F the mapping $\alpha \to \|y^\delta - F(x_\alpha^\delta)\|$ might not be continuous (although $\|y^\delta - F(x_\alpha^\delta)\|$ is monotonically increasing in α)[7]. To guarantee the existence of an α with (13), we need F to have some special properties.

2.1 Some properties of nonlinear operators

An operator F between Banach spaces is weakly sequentially closed if for every sequence $\{x_n\} \subset D(F)$

$$x_n \rightharpoonup x \text{ and } F(x_n) \rightharpoonup y \qquad \text{for } n \to \infty$$

implies $x \in D(F)$ and $F(x) = y$. F is called strongly continuous, if

$$x_n \rightharpoonup x \text{ implies } F(x_n) \to F(x) \ .$$

We might summarize some well known results about strongly continuous operators which will be needed later on.

Proposition 2.1 *Let X, Y, Z Banach spaces over $\mathbb{R}$.*

1. *If F is a linear compact operator, then F is strongly continuous.*
2. *If $F : Y \to Z$ is continuous and the embedding $X \hookrightarrow Y$ is compact, then $F : X \to Z$ is strongly continuous, and hence also weakly sequentially closed.*

For a proof, cf. [30], Propositions 21.29, 21.81 .
We might remark that the second case is common in ill posed problems when an operator F acting on Sobolev spaces can be decomposed into

$$F : H^{s+\varepsilon}(\Omega_1) \overset{i}{\hookrightarrow} H^s(\Omega_1) \overset{\tilde{F}}{\to} H^s(\Omega_2)$$

with $\varepsilon > 0$, bounded Ω_1, compact embedding operator i and continuous operator $\tilde{F}$. Throughout this paper we will therefore consider strongly continuous operators only.

2.2 Some results about the Tikhonov–functional

Let x_α^δ denote a minimizing element of the Tikhonov functional $J_\alpha(x)$. By the minimizing property of $x_{\alpha_1}^\delta$ we get for $\alpha_1 < \alpha_0$

$$J_{\alpha_1}(x_{\alpha_1}^\delta) \le J_{\alpha_1}(x_{\alpha_0}^\delta) \le J_{\alpha_0}(x_{\alpha_0}^\delta) \ . \tag{14}$$

It is well known that for general nonlinear operators F the mapping $\alpha \to \|y^\delta - F(x_\alpha^\delta)\|$ might be discontinuous. Due to the continuouity of F, the mapping $\alpha \to x_\alpha^\delta$ then has to be discontinuous too. Nevertheless, $\alpha \to J_\alpha(x_\alpha^\delta)$ is always continuous:

Proposition 2.2 *Let $\alpha_k \to \alpha > 0$ for $k \to \infty$, $\alpha_k > 0$ for all $k \in \mathbb{R}$. Then*

$$J_{\alpha_k}(x_{\alpha_k}^\delta) \to J_\alpha(x_\alpha^\delta) \quad \textit{for} \quad k \to \infty \ . \tag{15}$$

Proof:
We have

$$|J_{\alpha_k}(x_{\alpha_k}^\delta) - J_\alpha(x_\alpha^\delta)| = \begin{cases} J_{\alpha_k}(x_{\alpha_k}^\delta) - J_\alpha(x_\alpha^\delta) \text{ for } \alpha \le \alpha_k \\ J_\alpha(x_\alpha^\delta) - J_{\alpha_k}(x_{\alpha_k}^\delta) \text{ for } \alpha \ge \alpha_k \ , \end{cases}$$

and thus it follows from (14) for $\alpha \le \alpha_k$

$$\begin{aligned} J_{\alpha_k}(x_{\alpha_k}^\delta) - J_\alpha(x_\alpha^\delta) &\le J_{\alpha_k}(x_\alpha^\delta) - J_\alpha(x_\alpha^\delta) \\ &= (\alpha_k - \alpha)\|x_\alpha^\delta - \bar{x}\|^2 \end{aligned}$$

and for $\alpha \geq \alpha_k$

$$\begin{aligned} J_\alpha(x^\delta_\alpha) - J_{\alpha_k}(x^\delta_{\alpha_k}) &\leq J_\alpha(x^\delta_{\alpha_k}) - J_{\alpha_k}(x^\delta_{\alpha_k}) \\ &= (\alpha - \alpha_k)\|x^\delta_{\alpha_k} - \bar{x}\|^2 \ . \end{aligned}$$

With

$$\alpha_k\|x^\delta_{\alpha_k} - \bar{x}\|^2 \leq J_{\alpha_k}(x^\delta_{\alpha_k}) \leq J_{\alpha_k}(\bar{x}) = \|y^\delta - F(\bar{x})\|^2 \tag{16}$$

we get altogether

$$\begin{aligned} |J_{\alpha_k}(x^\delta_{\alpha_k}) - J_\alpha(x^\delta_\alpha)| &\leq |\alpha - \alpha_k| \cdot \max\left\{\frac{1}{\min\{\alpha_k\}}\|y^\delta - F(\bar{x})\|^2,\ \|x^\delta_\alpha - \bar{x}\|^2\right\} \\ &\to 0 \quad \text{for } k \to \infty \ . \end{aligned}$$

□

Now let us assume that no parameter α fulfilling the discrepancy principle (13) exists. Under reasonable assumptions to the a priori guess $\bar{x}$ in $J_\alpha(x)$ parameters $\alpha_0 > \alpha_1$ with

$$\|y^\delta - F(x^\delta_{\alpha_0})\| > c_1\delta \tag{17}$$

$$\|y^\delta - F(x^\delta_{\alpha_1})\| < \delta \tag{18}$$

exist:

Proposition 2.3 *Let $\bar{x}$ be chosen s.t.*

$$\|y^\delta - F(\bar{x})\| > c_1\delta \ , \tag{19}$$

If no regularization parameter α with (13) exists, then $\alpha_0 > \alpha_1$ with (17), (18) can be found.

Proof:
We have

$$\begin{aligned} \|\bar{x} - x^\delta_{\alpha_0}\|^2 &\leq \frac{1}{\alpha_0}\left(\|y^\delta - F(x^\delta_{\alpha_0})\|^2 + \alpha_0\|\bar{x} - x^\delta_{\alpha_0}\|^2\right) \\ &\leq \frac{1}{\alpha_0}\|y^\delta - F(\bar{x})\|^2 \ . \end{aligned}$$

It then follows with $y^\delta \neq F(\bar{x})$ that $x^\delta_{\alpha_0}$ converges to $\bar{x}$ for $\alpha_0 \to \infty$ and, due to the continuouity of F,

$$\|y^\delta - F(x^\delta_{\alpha_0})\| \to \|y^\delta - F(\bar{x})\| \ .$$

Because of (19), $\|y^\delta - F(x^\delta_{\alpha_0})\| > c_1\delta$ must hold for α_0 big enough. On the other hand, we can estimate $\|y^\delta - F(x^\delta_{\alpha_1})\|$ from above by

$$\begin{aligned} \|y^\delta - F(x^\delta_{\alpha_1})\|^2 &\leq \|y^\delta - F(x^\delta_{\alpha_1})\|^2 + \alpha_1\|\bar{x} - x^\delta_{\alpha_1}\|^2 \\ &\leq \delta^2 + \alpha_1\|\bar{x} - x_*\|^2 \ , \end{aligned}$$

which means for small α_1 either $\|y^\delta - F(x^\delta_{\alpha_1})\| < \delta$ or $\delta \leq \|y^\delta - F(x^\delta_\alpha)\| \leq c_1\delta$. Because we have assumed that no parameter with (13) exists, we get $\|y^\delta - F(x^\delta_{\alpha_1})\| < \delta$.

□

Assumption (19) is quite natural: Indeed, if $\|y^\delta - F(\bar{x})\| \leq c_1\delta$ holds, then $\bar{x}$ would be taken as approximation to the solution x_*.
If no parameter α with (13) exists, then $\|y^\delta - F(x^\delta_\alpha)\|$ has a jump at a certain parameter $\tilde{\alpha}$. Because $J_\alpha(x^\delta_\alpha)$ is continuous, $\alpha\|x^\delta_\alpha - \bar{x}\|$ must have a jump of the same size.

Proposition 2.4 *Assume that no parameter α with (13) exists and (19) holds. Then there exists a parameter $\tilde{\alpha}$ such that (17), (18) holds for all $\alpha_1 < \tilde{\alpha} < \alpha_0$, and α_0, α_1 arbitrary close to $\tilde{\alpha}$. Moreover, we get*

$$\|x^\delta_{\alpha_1} - x^\delta_{\alpha_0}\| \geq \frac{(c_1^2-1)\delta^2}{4\alpha_0\|x^\delta_{\alpha_1} - \bar{x}\|} \ , \tag{20}$$

Proof:
Proposition 2.3 ensures the existence of $\alpha_{u,1} > \alpha_{l,1}$ with $\|y^\delta - F(x_{\alpha_{l,1}})\| < \delta < c_1\delta < \|y^\delta - F(x_{\alpha_{u,1}})\|$. Setting $\alpha_{m,j} = \alpha_{l,j} + \frac{\alpha_{u,j}-\alpha_{l,j}}{2}$, $j = 1, 2, ..$, we have either $\|y^\delta - F(x_{\alpha_{m,j}})\| < \delta$ or $\|y^\delta - F(x_{\alpha_{m,j}})\| > c_1\delta$; in the first case we set $\alpha_{l,j+1} = \alpha_{m,j}$, $\alpha_{u,j+1} = \alpha_{u,j}$, in the second case $\alpha_{l,j+1} = \alpha_{l,j}$ and $\alpha_{u,j+1} = \alpha_{m,j}$. According to the construction of both sequences, $\{\alpha_{l,j}\}$ and $\{\alpha_{u,j}\}$ converge to the same limit point, $\alpha_{l,j} \uparrow \tilde{\alpha}$, $\alpha_{u,j} \downarrow \tilde{\alpha}$ for $j \to \infty$. Due to Proposition 2.2, we can especially choose α_0, α_1 with $\alpha_1 < \tilde{\alpha} < \alpha_0$ and (17), (18) such that

$$J_{\alpha_1}(x^\delta_{\alpha_1}) = J_{\alpha_0}(x^\delta_{\alpha_0}) - \eta \tag{21}$$

and $\eta < \frac{\gamma\delta^2}{2}$, $\gamma = c_1^2 - 1$ hold. From the definition of $J_\alpha(x)$, (17), (18) and (21) it then follows

$$\begin{aligned}
\alpha_1\|x^\delta_{\alpha_1} - \bar{x}\|^2 - \alpha_0\|x^\delta_{\alpha_0} - \bar{x}\|^2 &= \|y^\delta - F(x^\delta_{\alpha_0})\|^2 - \|y^\delta - F(x^\delta_{\alpha_1})\|^2 - \eta \\
&\geq c_1^2\delta^2 - \delta^2 - \eta = \gamma\delta^2 - \eta \\
&\geq \frac{\gamma}{2}\delta^2 \ ,
\end{aligned}$$

and because of $\alpha_0 > \alpha_1$

$$\alpha_0(\|x^\delta_{\alpha_1} - \bar{x}\|^2 - \|x^\delta_{\alpha_0} - \bar{x}\|^2) > \alpha_1\|x^\delta_{\alpha_1} - \bar{x}\|^2 - \alpha_0\|x^\delta_{\alpha_0} - \bar{x}\|^2 \geq \frac{\gamma}{2}\delta^2$$

or

$$\|x^\delta_{\alpha_1} - \bar{x}\|^2 - \|x^\delta_{\alpha_0} - \bar{x}\|^2 \geq \frac{\gamma\delta^2}{2\alpha_0} > 0 \ . \tag{22}$$

Using (22) and

$$\begin{aligned}
\|x^\delta_{\alpha_1} - \bar{x}\|^2 - \|x^\delta_{\alpha_0} - \bar{x}\|^2 &= (\|x^\delta_{\alpha_1} - \bar{x}\| - \|x^\delta_{\alpha_0} - \bar{x}\|)(\|x^\delta_{\alpha_1} - \bar{x}\| + \|x^\delta_{\alpha_0} - \bar{x}\|) \\
&\leq 2\|x^\delta_{\alpha_1} - \bar{x}\|(\|x^\delta_{\alpha_1} - \bar{x}\| - \|x^\delta_{\alpha_0} - \bar{x}\|) \\
&\leq 2\|x^\delta_{\alpha_1} - \bar{x}\|\|x^\delta_{\alpha_1} - x^\delta_{\alpha_0}\| \ ,
\end{aligned}$$

we arrive at

$$\|x^\delta_{\alpha_1} - x^\delta_{\alpha_0}\| \geq \frac{\gamma\delta^2}{4\alpha_0\|x^\delta_{\alpha_1} - \bar{x}\|} ,$$

which shows that the distance between $x^\delta_{\alpha_1}$ and $x^\delta_{\alpha_0}$ for arbitrary close parameters $\alpha_0 > \alpha_1$ with (17), (18) is bounded from below.

□

In the following we will show that strong continuouity of F ensures the existence of a regularization parameter with (13).

Proposition 2.5 *Let $\{\alpha_k\}_{k\in\mathbb{N}}$, $\alpha_k > 0$ for all $k \in \mathbb{N}$ be a sequence with $\alpha_k \to \alpha > 0$ for $k \to \infty$, and $x^\delta_{\alpha_k}$ a corresponding minimizing element of the Tikhonov–functional $J_{\alpha_k}(x)$. If F is strongly continuous, then there exists a weakly convergent subsequence $x^\delta_{\alpha_{kn}}$ of $x^\delta_{\alpha_k}$,*

$$x^\delta_{\alpha_{kn}} \rightharpoonup \tilde{x} ,$$

and $\tilde{x}$ is a minimizing element of $J_\alpha(x)$.

Proof:
As in (16) we get

$$\|x^\delta_{\alpha_k} - \bar{x}\|^2 \leq \frac{1}{\min\{\alpha_k\}}\|y^\delta - F(\bar{x})\|^2 ,$$

i.e. $x^\delta_{\alpha_k}$ is bounded. Thus, there exists a weakly convergent subsequence of $\{x^\delta_{\alpha_k}\}$ (for simplicity of notation, this subsequence will be denoted by $x^\delta_{\alpha_k}$ again), $x^\delta_{\alpha_k} \rightharpoonup \tilde{x}$ and

$$\|\bar{x} - \tilde{x}\| \leq \liminf \|\bar{x} - x^\delta_{\alpha_k}\| .$$

Moreover, due to the strong continuouity of F, we observe

$$y^\delta - F(x^\delta_{\alpha_k}) \longrightarrow y^\delta - F(\tilde{x}) ,$$

and according to Proposition 2.2 we get

$$J_{\alpha_k}(x^\delta_{\alpha_k}) \to J_\alpha(x^\delta_\alpha) .$$

Altogether this yields

$$\begin{aligned} J_\alpha(\tilde{x}) &= \lim_{k\to\infty} \|y^\delta - F(x^\delta_{\alpha_k})\|^2 + \alpha\|\tilde{x} - \bar{x}\|^2 \\ &\leq \liminf \|y^\delta - F(x^\delta_{\alpha_k})\|^2 + \liminf \Big((\alpha - \alpha_k + \alpha_k)\|\bar{x} - x^\delta_{\alpha_k}\|^2\Big) \\ &\leq \liminf \Big(\|y^\delta - F(x^\delta_{\alpha_k})\|^2 + \alpha_k\|\bar{x} - x^\delta_{\alpha_k}\|^2 + (\alpha - \alpha_k)\|\bar{x} - x^\delta_{\alpha_k}\|^2\Big) \\ &= \lim_{k\to\infty} J_{\alpha_k}(x^\delta_{\alpha_k}) = J_\alpha(x^\delta_\alpha) . \end{aligned}$$

Now we have shown $J_\alpha(\tilde{x}) \le J_\alpha(x_\alpha^\delta)$ and hence $\tilde{x}$ is a minimizer of $J_\alpha(x)$.

□

We might note that the minimizing element of $J_\alpha(x)$ does not have to be unique; but we are only interested in finding a weakly convergent subsequence to an arbitrary minimizing element, which means we can assign $\tilde{x}$ to x_α^δ. On the other hand, if $J_\alpha(x)$ has a unique minimizer, then every subsequence of $x_{\alpha_k}^\delta$ has a subsequence which converges weakly to to the unique minimizer x_α^δ, and it follows by the convergence principles that the sequence $x_{\alpha_k}^\delta$ converges itself weakly to x_α^δ.

Theorem 2.6 *Let F be a strongly continuous operator and $\bar{x}$ be chosen such that (19) holds. Then there exists a parameter α for Tikhonov–regularization s.t.*

$$\delta \le \|y^\delta - F(x_\alpha^\delta)\| \le c_1\delta \tag{23}$$

holds.

Proof:
Let us assume there exists no such parameter. We set

$$M := \{\alpha : \|y^\delta - F(x_\alpha^\delta)\| < \delta\}$$

and $\bar{\alpha} = \sup M$. According to Proposition 2.3, $0 < \bar{\alpha} < \infty$ holds. We have to consider two cases:

1. $\bar{\alpha} \in M$. Then we choose a sequence $\alpha_k \downarrow \bar{\alpha}$. Due to Proposition 2.5 we can find a subsequence $\{x_{\alpha_{k_n}}^\delta\}$ of $\{x_{\alpha_k}^\delta\}$ with $x_{\alpha_{k_n}}^\delta \rightharpoonup x_{\bar{\alpha}}^\delta$. It is $\alpha_{k_n} > \alpha$, and because no parameter with (13) exists ,

$$\|y^\delta - F(x_{\alpha_{k_n}}^\delta)\| > c_1\delta \tag{24}$$

must hold for all α_{k_n}. But due to the strong continuouity of F we observe

$$\|y^\delta - F(x_{\alpha_{k_n}}^\delta)\| \to \|y^\delta - F(x_{\bar{\alpha}}^\delta)\| < \delta\ , \tag{25}$$

which is a contradiction to (24).
2. $\bar{\alpha} \notin M$. Here we choose $\alpha_k \uparrow \bar{\alpha}$, and we can find a subsequence $\{x_{\alpha_{k_n}}^\delta\}$ of $\{x_{\alpha_k}^\delta\}$ with $x_{\alpha_{k_n}}^\delta \to x_{\bar{\alpha}}$. For all k_n holds then the inequality

$$\|y^\delta - F(x_{\alpha_{k_n}}^\delta)\| < \delta \tag{26}$$

and $\|y^\delta - F(x_{\alpha_{k_n}}^\delta)\| \to \|y^\delta - F(x_{\bar{\alpha}}^\delta)\| > c_1\delta$ for $k_n \to \infty$, which is a contradiction to (26).

□

For the numerical realization of Morozov's discrepancy principle, we can now propose the well known iterative algorithm from linear inverse problems:

- choose $c_1 > 1$, $0 < q_0 < 1$ and α_0 with (17)
- Set $\alpha_u = \alpha_0$, j=0
- while not $(\delta \le \|y^\delta - F(x^\delta_{\alpha_j})\| \le c_1\delta)$
  - ◇ If $\|y^\delta - F(x^\delta_{\alpha_{j-1}})\| > c_1\delta$ then $q_j = q_{j-1}$, $\alpha_j = q_j\alpha_{j-1}$
    else $q_j = q_{j-1} + (1 - q_{j-1})/2$, $\alpha_j = q_j\alpha_u$
  - ◇ Compute $x^\delta_{\alpha_j}$.
  - ◇ If $\|y^\delta - F(x^\delta_{\alpha_j})\| > c_1\delta$ then $\alpha_u = \alpha_j$

  end

The algorithm produces a monotone decreasing sequence of regularization parameters α_j at the beginning. Only if the norm of the residual jumps behind the trust region, $\|y^\delta - F(x^\delta_{\alpha_{j-1}})\| < \delta < c_1\delta < \|y^\delta - F(x^\delta_{\alpha_j})\|$, a bigger parameter is used. However, if q is chosen close to 1 or if c_1 is reasonable big, such a jump will not occur in a numerical realization. In [28] it was outlined that Morozov's discrepancy principle yields sometimes a to small regularization parameter; this can be avoided by choosing c_1 big enough.
When an iterative algorithm for minimizing $J_\alpha(x)$ is used, then the computational effort might depend on a good starting value for the iteration. For $\alpha_{j+1} = \alpha_j \cdot q$, $q < 1$, one would like to take the already computed minimizing element of $J_{\alpha_j}(x)$, $x^\delta_{\alpha_j}$, as starting value for the iteration for minimizing $J_{\alpha_{j+1}}(x)$. For $q \approx 1$ this can be justified if the mapping $\alpha \to x^\delta_\alpha$ is continuous. We will now show that for every sequence $\alpha_k \to \alpha$ exists at least a convergent subsequence s.t. $x^\delta_{\alpha_{kn}} \to x^\delta_\alpha$ holds. For this result we have to employ a knew property of the operator F. We require

$$x_n \rightharpoonup x \text{ for } n \to \infty \Longrightarrow F'(x_n)^* z \to F'(x)^* z \text{ for } n \to \infty\ . \tag{27}$$

In the next section we will give several examples where this condition is fulfilled.

Theorem 2.7 *Let the assumptions of Proposition 2.5 hold. If the Frechét derivative F' of F is Lipschitz–continuous,*

$$\|F'(x) - F'(z)\| \le L\|x - z\|\ , \tag{28}$$

and in addition condition (27) holds, then there exists a convergent subsequence $\{x^\delta_{\alpha_{kn}}\}$ of $\{x^\delta_{\alpha_k}\}$,

$$x^\delta_{\alpha_{kn}} \to \tilde{x}\ ,$$

and $\tilde{x}$ is a minimizer of $J_\alpha(x)$. If in addition $J_\alpha(x)$ has a unique minimizer, then the whole sequence converges to the minimizer of $J_\alpha(x)$.

Proof:
According to Proposition 2.5, we can find a weakly convergent subsequence $\{x^\delta_{\alpha_{kn}}\}$ of $\{x^\delta_{\alpha_k}\}$

with weak limit $\tilde{x}$, where $\tilde{x}$ is a minimizer of $J_\alpha(x)$. For simplicity of notation, we will again denote $x^\delta_{\alpha_{k_n}}$ by $x^\delta_{\alpha_k}$. The necessary conditions for a minimum of $J_\alpha(x)$ and $J_{\alpha_k}(x)$ are

$$\begin{aligned} F'(x)^*(y^\delta - F(x)) - \alpha(x-\bar{x}) &= 0 \\ F'(x)^*(y^\delta - F(x)) - \alpha_k(x-\bar{x}) &= 0 \; . \end{aligned}$$

It then follows

$$\alpha_k x^\delta_{\alpha_k} - \alpha x^\delta_\alpha - (\alpha_k - \alpha)\bar{x} = F'(x^\delta_{\alpha_k})^*(y^\delta - F(x^\delta_{\alpha_k})) - F'(x^\delta_\alpha)^*(y^\delta - F(x^\delta_\alpha)) \tag{29}$$

The left hand side of (29) can be rewritten as

$$\alpha_k x^\delta_{\alpha_k} - \alpha x^\delta_\alpha - (\alpha_k - \alpha)\bar{x} = (\alpha_k - \alpha)x^\delta_{\alpha_k} + \alpha(x^\delta_{\alpha_k} - x^\delta_\alpha) - (\alpha_k - \alpha)\bar{x} \; .$$

We have already shown that $\|x^\delta_{\alpha_k}\|$ is bounded and therefore $\|(\alpha_k - \alpha)x^\delta_{\alpha_k}\| \to 0$ as well as $\|(\alpha_k - \alpha)\bar{x}\| \to 0$ for $k \to \infty$. In order to prove the strong convergence of $x^\delta_{\alpha_k}$ to x^δ_α it is sufficient to show that the right hand side of (29) converges to zero. By setting $y_\alpha := y^\delta - F(x^\delta_\alpha)$ and $y_{\alpha_k} = y^\delta - F(x^\delta_{\alpha_k})$ we have

$$F'(x^\delta_{\alpha_k})^*(y_{\alpha_k}) - F'(x^\delta_\alpha)^*(y_\alpha) = F'(x^\delta_{\alpha_k})^*(y_{\alpha_k} - y_\alpha) + (F'(x^\delta_{\alpha_k}) - F'(x^\delta_\alpha))^*(y_\alpha)$$

and

$$\|F'(x^\delta_{\alpha_k})^*(y_{\alpha_k} - y_\alpha)\| \le \|F'(x^\delta_{\alpha_k})\| \|y_{\alpha_k} - y_\alpha\| \; .$$

By the Lipschitz–continuouity of F', the norm of $F'(x^\delta_{\alpha_k})$ is uniformly bounded:

$$\begin{aligned} \|F'(x^\delta_{\alpha_k})\| &\le \|F'(x^\delta_{\alpha_k}) - F'(x^\delta_\alpha)\| + \|F'(x^\delta_\alpha)\| \\ &\le L\|x^\delta_{\alpha_k} - x^\delta_\alpha\| + \|F'(x^\delta_\alpha)\| < C \; . \end{aligned}$$

Due to the weak convergence of $x^\delta_{\alpha_k}$ to x^δ_α we conclude

$$\|y_{\alpha_k} - y_\alpha\| \to 0 \text{ for } k \to \infty$$

and so $\|F'(x^\delta_{\alpha_k})^*(y_{\alpha_k} - y_\alpha)\| \to 0$.
Condition (27) yields

$$F'(x^\delta_{\alpha_k})^*(y_\alpha) \to F'(x^\delta_\alpha)^*(y_\alpha) \;\text{ for } k \to \infty$$

and thus we have shown

$$F'(x^\delta_{\alpha_k})^*(y_{\alpha_k}) - F'(x^\delta_\alpha)^*(y_\alpha) \to 0 \; .$$

After the proof of Proposition 2.5 we have seen that in case of a unique minimizer of $J_\alpha(x)$, the whole sequence $x^\delta_{\alpha_k}$ did weakly converge to $\tilde{x}$. With the above arguments the convergence is also strongly.

□

At the end of this section we would like to give a convergence and a convergence rate result.

Theorem 2.8 *Let $F : X \to Y$ be a completely continuous operator and $\delta_k \to 0$ for $k \to \infty$. If y^{δ_k} denotes data with $\|y^{\delta_k} - y\| \le \delta_k$ and $x_{\alpha_k}^{\delta_k}$ is a minimizer of the Tikhonov–functional (5) with y^δ replaced by y^{δ_k} and the parameter α_k chosen by Morozov's discrepancy principle, then $x_{\alpha_k}^{\delta_k}$ has a convergent subsequence. The limit of every convergent subsequence is an $\bar{x}$–minimum–norm–solution of (1). If, in addition, the $\bar{x}$– minimum–norm–solution x_* of (1) is unique, then*

$$x_{\alpha_k}^{\delta_k} \to x_* \text{ for } k \to \infty \ .$$

For proof, we refer to Proposition 3.5 in [28].
In general, the convergence might be arbitrary slow. It is therefore of interest to have a convergence rate result.

Theorem 2.9 *Let $F : X \to Y$ be a completely continuous operator with convex definition area $D(F)$ and let x_* be a $\bar{x}$–minimum–norm–solution of $F(x) = y$ and $\|y - y^\delta\| \le \delta$. Assume that (19) and (28) hold. Moreover, we require the following range conditions:*

1. *there exists $\omega \in Y$ satisfying*
$$x_* - \bar{x} = F'(x_*)^* \omega \tag{30}$$
and
2. $L\|\omega\| < 1$.

If the regularization parameter α is chosen s.t.

$$\delta \le \|y^\delta - F(x_\alpha^\delta)\| \le c_1 \delta \tag{31}$$

holds, then we obtain

$$\|x_\alpha^\delta - x_*\| \le \left(\frac{2(1+c_1)\|\omega\|}{1 - L\|\omega\|} \right)^{1/2} \sqrt{\delta} \ . \tag{32}$$

Proof:
Theorem 2.6 ensures the existence of a parameter α with (31). The proof is a modification of a convergence proof for an a priori parameter choice for Tikhonov regularization. As in [10], p.246 we obtain

$$\|y^\delta - F(x_\alpha^\delta)\|^2 + \alpha\|x_\alpha^\delta - x_*\|^2 \le \delta^2 + 2\alpha\|\omega\|\delta + 2\alpha\|\omega\|\|y^\delta - F(x_\alpha^\delta)\| + \alpha L\|\omega\|\|x_\alpha^\delta - x_*\|^2 \tag{33}$$

or

$$\alpha(1 - L\|\omega\|)\|x_\alpha^\delta - x_*\|^2 \le \delta^2 - \|y^\delta - F(x_\alpha^\delta)\|^2 + 2\alpha\|\omega\|(\delta + \|y^\delta - F(x_\alpha^\delta)\|) \ , \tag{34}$$

and because of (31) we have $\delta^2 - \|y^\delta - F(x_\alpha^\delta)\|^2 \le 0$ and $\delta + \|y^\delta - F(x_\alpha^\delta)\| \le (1+c_1)\delta$. Altogether we arrive at

$$\|x_\alpha^\delta - x_*\|^2 \le \frac{2(1+c_1)\|\omega\|}{1 - L\|\omega\|}\delta \ .$$

□

3 Examples

In the following we will give some examples which will meet the conditions of Section 2. For the autoconvolution operator it is shown that the operator is strongly continuous and meets (27) but not Scherzer's conditions (12). Other examples come from bilinear operator equations and medical imaging.

3.1 The autoconvolution operator

We consider the operator

$$\tilde{F}(x)(s) := (x * x)(s) = \int_{\mathbb{R}} x(s-t)x(t)\,dt \tag{35}$$

For $x \in L^2[a,b],\ \ -\infty < a < b < \infty$, we get by setting $x(t) = 0$ for $t \notin [a,b]$ that $\mathrm{supp}\tilde{F}(x)(s) \subset [c,d],\ \ -\infty < c < d < \infty$, and we can consider

$$\tilde{F} : L^2[a,b] \longrightarrow L^2[c,d] \ .$$

Hölders inequality yields

$$\|\tilde{F}(x)\|_{L^2[c,d]} \le (d-c)^{1/2} \|x\|^2_{L^2[a,b]} \ .$$

The operator $\tilde{F}$ is especially a (symmetric) bilinear operator, $\tilde{F}(x) = B(x,x)$ with $\|B\| = (d-c)^{1/2}$, and it follows immediately that $\tilde{F}$ is Frechét differentiable with derivative

$$\tilde{F}'(x)h = 2B(x,h) \ . \tag{36}$$

To obtain a weakly sequentially closed operator, we have to assume some smoothness of the solution of the equation $\tilde{F}(x) = y$. Therefore we consider the autoconvolution operator between a Sobolev space of order $\alpha > 0$ and $L^2[c,d]$: We define F by

$$F:\ H_0^\alpha[a,b] \overset{i}{\hookrightarrow} L^2[a,b] \overset{\tilde{F}}{\to} L^2[c,d] \ , \tag{37}$$

where i denotes the compact embedding from $H_0^\alpha[a,b]$ to $L^2[a,b]$. From Proposition 2.1 follows that F is strongly continuous and hence weakly sequentially closed.

Proposition 3.1 *The Frechét derivative F' of F is Lipschitz continuous with*

$$\begin{aligned} \|F'(x) - F'(z)\|_{H_0^\alpha[a,b]\to L^2[c,d]} &\le 2(d-c)^{1/2}\|x-z\|_{L^2[a,b]} && (38)\\ &\le 2(d-c)^{1/2}\|x-z\|_{H_0^\alpha[a,b]} && (39)\end{aligned}$$

Proof:
We have $(F'(x) - F'(z))h = 2(x-z) * h$ and by Hölders inequality

$$\begin{aligned} \frac{1}{4}\|(F'(x) - F'(z))h\|^2_{L^2[c,d]} &= \int_c^d \left| \int_a^b (x-z)(s-t)h(t)\,dt \right|^2 ds \\ &\le (d-c)\|x-z\|^2_{L^2[a,b]}\|h\|^2_{L^2[a,b]} \\ &\le (d-c)\|x-z\|^2_{L^2[a,b]}\|h\|^2_{H_0^\alpha[a,b]} \\ &\le (d-c)\|x-z\|^2_{H_0^\alpha[a,b]}\|h\|^2_{H_0^\alpha[a,b]} \ , \end{aligned}$$

and the last two inequalities prove the Lipschitz continuouity.

□

A by-product of (38) is

Proposition 3.2 *Let* $x_n \rightharpoonup x$ *in* $H_0^\alpha[a,b]$ *for* $n \to \infty$. *Then*

$$\|F'(x_n) - F'(x)\|_{H_0^\alpha[a,b]\to L^2[c,d]} \to 0 \text{ for } n \to \infty \ . \tag{40}$$

Proof:
From $x_n \rightharpoonup x$ in $H_0^\alpha[a,b]$ follows $x_n \to x$ in $L^2[a,b]$, and thus from (38) the proposition.

□

As a consequence, $\|F'(x_n)^* - F'(x)^*\| \to 0$ and we have finally shown that (27) holds. Morozov's discrepancy principle can be used as parameter choice for Tikhonov regularization of the autoconvolution operator, and Theorem 2.7 applies. Our results were given in one dimension only, but they easily extend to higher dimensions. Additionally, we might remark that condition (27), which was the only new condition to the operator F, was a consequence of the Lipschitz–continuouity of F'.

We will now see that the conditions (11), (12) from [27] will usually not hold. In case of the autoconvolution operator, (11) reads

$$(x - z) * v = z * k \ . \tag{41}$$

Using the Fourier transform gives $\widehat{(x-z)} \cdot \widehat{v} = \widehat{z} \cdot \widehat{k}$. Formally, k is then given by

$$\widehat{k} = \frac{\widehat{(x-z)} \cdot \widehat{v}}{\widehat{z}} \ . \tag{42}$$

If $|\widehat{z}(\omega)| > c > 0$ holds, then condition (12) will hold. But whenever $\widehat{z}(\omega)$ has zeros, then z might not even belong to the proper function space. To illustrate this, let us assume the autoconvolution operator between $H_0^\alpha[a,b]$ and $L^2[c,d]$ with $\alpha > 1/2$. Then k has to belong to $H_0^{1/2}[a,b]$, especially $k \in L^1[a,b]$ and it follows that $\widehat{k}$ is a continuous and bounded function. The functions x and v belong to $H_0^{1/2}[a,b]$, and $\widehat{x} \cdot \widehat{v}$ is continuous and bounded too. We choose x, v, z and ω_0 in such a way, that $\widehat{z}(\omega_0) = 0$ and $\widehat{x} \cdot \widehat{v}(\omega_0) \neq 0$. For a sequence $\omega_n \to \omega_0$ follows $|\widehat{k}(\omega_n)| \to \infty$, which means that $\widehat{k}$ is not bounded and therefore does not belong to $H_0^{1/2}[a,b]$. As a consequence, condition (12) is violated.

3.2 Bilinear operator equation

Of great interest are operators which can be decomposed into

$$\tilde{F}(x) = Af + B(f, \mu) \ , \tag{43}$$

with $x = (f, \mu) \in X_1 \times X_2$, X_1, X_2 Hilbert spaces, A a continuous linear operator in f and B a bilinear operator in (f, μ):

$$\begin{aligned} A &: X_1 \to Y && (44) \\ B &: X_1 \times X_2 \to Y && (45) \end{aligned}$$

$$\begin{aligned} B(\lambda(f_1 + f_2), \mu) &= \lambda(B(f_1, \mu) + B(f_2, \mu)) && (46) \\ B(f, \lambda(\mu_1 + \mu_2)) &= \lambda(B(f, \mu_1) + B(f, \mu_2)) && (47) \\ \|B(f, \mu)\| &\leq \|B\|\|f\|\|\mu\| \ . && (48) \end{aligned}$$

The function space $X_1 \times X_2$ is turned into an Hilbert space by setting

$$\langle (f, \mu), (g, \nu) \rangle_{X_1 \times X_2} := \langle f, g \rangle_{X_1} + \langle \mu, \nu \rangle_{X_2} \ . \tag{49}$$

Operators of type (43) occur in parameter estimation problems for partial differential operators [6, 9, 16, 24, 17] and in the area of medical imaging (compare next section).

It is easy to see that $\tilde{F}$ is Frechét differentiable with derivative

$$\tilde{F}'(f, \mu)(h_1, h_2) = Ah_1 + B(h_1, \mu) + B(f, h_2) \ , \tag{50}$$

and

$$\begin{aligned} \|(\tilde{F}'(f, \mu) - \tilde{F}'(g, \nu))(h_1, h_2)\| &= \|B(h_1, \mu) + B(f, h_2) - B(h_1, \nu) - B(g, h_2)\| \\ &= \|B(h_1, \mu - \nu) + B(f - g, h_2)\| \\ &\leq \|B\|\|\mu - \nu\|\|h_1\| + B\|f - g\|\|h_2\| \\ &\leq 2\|B\|\|(f - g, \mu - \nu)\|\|(h_1, h_2)\| \ . \end{aligned}$$

Therefore $\tilde{F}'$ is Lipschitz continuous with constant $L = 2\|B\|$. It remains to show that $\tilde{F}$ is strongly continuous and that $\tilde{F}'(f_n, \mu_n)(h_1, h_2) \to \tilde{F}'(f, \mu)(h_1, h_2)$ if $(f_n, \mu_n) \rightharpoonup (f, \mu)$ for $n \to \infty$ holds. In applications it might happen that $\tilde{F}$ already meets these conditions as operator from $X_1 \times X_2$ to Y. If not, this can be achieved by assuming more "regularity" of the solution of $\tilde{F}(x) = y$, which means we have to change the definition area of $\tilde{F}$. Let us assume that there exist function spaces X_1^s and X_2^s, and compact embedding operators $i_1^s : X_1^s \to X_1$, $i_2^s : X_2^s \to X_2$. Then we can consider

$$F : X_1^s \times X_2^s \stackrel{i_1^s \otimes i_2^s}{\longrightarrow} X_1 \times X_2 \stackrel{\tilde{F}}{\longrightarrow} Y \tag{51}$$

and get from Proposition 2.1 (2) that F is strongly continuous and weakly sequentially closed. Now, exactly as for the autoconvolution operator, we obtain

$$\begin{aligned} \|F'(f, \mu) - F'(g, \nu)\|_{X_1^s \times X_2^s \to Y} &\leq L\|(f, \mu) - (g, \nu)\|_{X_1 \times X_2} && (52) \\ &\leq L\|(f, \mu) - (g, \nu)\|_{X_1^s \times X_2^s} && (53) \end{aligned}$$

If $(f_n, \mu_n) \rightharpoonup (f, \mu)$ in $X_1^s \times X_2^s$, then $(f_n, \mu_n) \to (f, \mu)$ in $X_1 \times X_2$ and (52) yields $F'(f_n, \mu_n) \to F'(f, \mu)$ and $F'(f_n, \mu_n)^* \to F'(f, \mu)^*$ in the operator norm. This shows that Morozov's discrepancy principle is applicable and Theorem 2.7 holds.

We might remark that our argument will apply to arbitrary nonlinear continuous and Frechét differentiable operators $F : X \to Y$ with Lipschitz continuous derivative as long as a function space X^s with compact embedding to X is available. A common choice for X^s might be a Sobolev space over a bounded region Ω_1 and for X the space $L^2(\Omega_2)$.

3.3 Single Photon Emission Computerized Tomography (SPECT)

Some of the most challenging ill–posed problems arise in the area of medical imaging. In SPECT, one tries to reconstruct the distribution of a radiopharmaceutical inside a human body by measuring the intensity of the radiation outside the body. As the name suggests, SPECT is related to the Computerized Tomography (CT), where one has to reconstruct the density of a body by measuring the outcoming intensity of X–rays through the body. In contrast to CT, where the measured intensity depends only on the intensity of the incoming X–ray and the density μ of the tissue along the path of the X–ray, depend the measurements for SPECT on the activity function f (which describes the distribution of the radiopharmaceutical) and the density μ of the tissue. The measured data y and the tuple (f, μ) are linked by the Attenuated Radon Transform (ATRT),

$$y = R(f, \mu)(s, \omega) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) e^{-\int_t^\infty \mu(s\omega^\perp + \tau\omega)\, d\tau}\, dt\ , \tag{54}$$

$s \in \mathbb{R},\ \omega \in S^1$. As for the Radon Transform, the data are represented as line integrals over all possible unit vectors ω. Usually both f and μ are unknown functions, and R is a nonlinear operator. During the last decade several papers on this problem were published [4, 20, 21, 23, 29]. Dicken [8] examined the mapping properties of the ATRT and concluded that under some reasonable assumptions to the smoothness of f and μ Tikhonov regularization with a priori parameter choice is applicable to regularize (54). In [26] a bilinear approximation $\tilde{R}$ to R was introduced:

$$\tilde{R}(f, \tilde{\mu}) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) e^{-\int_t^\infty \mu_0(s\omega^\perp + \tau\omega)\, d\tau} (1 - \int_t^\infty \tilde{\mu}(s\omega^\perp + \tau\omega) d\tau)\, dt\ . \tag{55}$$

In this approximation the exponential term in (54) was simply replaced by the first two terms of its Taylor expansion around a guess μ_0 for the attenuation function $\mu = \mu_0 + \tilde{\mu}$. Moreover, iterative methods for solving $y = \tilde{R}(f, \tilde{\mu})$ were proposed in this paper. In the following, it shall be shown that Morozov's discrepancy principle for Tikhonov regularization can be applied to SPECT. The analysis will be done for the ATRT operator only; for the bilinearized version (55) a similar result yields.
To get the desired results like strong continuouity or Frechét differentiability for the ATRT operator, it has to be considered between proper function spaces. Additionally, there will be some trouble with the unbounded growth of the exponential function for negative arguments. In the following, we will summarize some results from [8]. Dicken introduces an operator R_ϱ

by

$$R_\varrho(f,\mu)(s,\omega) = \int_{\mathbb{R}} f(s\omega^\perp + t\omega) E(-\int_{-\rho}^{\rho} \mu(s\omega^\perp + \tau\omega)\, d\tau)\, dt \ . \tag{56}$$

The function $E \in C^2(\mathbb{R})$ is chosen such that

$$E(x) = exp(-x) \text{ for } x \in \mathbb{R}^+$$

and $|E|$, $|E'|$ and $|E''|$ are bounded. For SPECT, the functions f and μ will be nonnegative with compact support. If we assume that f has its support in a disc with radius ϱ, then the operator R_ϱ coincides with R for admissible sets (f,μ). Fortunately, R_ϱ has much better mapping properties than R. If the definition area $D(R_\varrho)$ is given by

$$D(R_\varrho) := D_{s_1,s_2,C} = \{(f,\mu) \in H_0^{s_1} \times H_0^{s_2} \mid \|f\|_\infty \le C\} \ , \tag{57}$$

then the following Proposition holds:

Proposition 3.3 *Let $R_\varrho : D_{s_1,s_2,C} \to L^2(S^1 \times [-\varrho,\varrho])$. If $s_1, \ s_2 > \frac{2}{5}$, then R_ϱ is a strongly continuous. The Frechét derivative exists for all $(f,\mu) \in D_{s_1,s_2,C}$ and is Lipschitz continuous.*

For a proof, cnf. Theorem 4.10 in [8]. Strong continuouity of R_ϱ follows from the decomposition of F into a linear compact and an continuous operator. There is even a more detailed version of the above Proposition given, with more possible combinations of $s_1, \ s_2$ such that the Proposition still holds. An important point is to choose $s_1 < 1/2$, because for our medical application the activity function f will usually not belong to $H_0^{s_1}$ for $s_1 \ge 1/2$.

Theorem 3.4 *Let the conditions of Proposition 3.3 hold. Then condition (27) holds.*

Proof:
Let $s_1, \ s_2 > 2/5$ be given. According to Proposition 3.3, the Frechét derivative of F is Lipschitz continuous for every $\bar{s}_1, \ \bar{s}_2 > 2/5$. Thus we can find $\bar{s}_1, \ \bar{s}_2$ with

$$\frac{2}{5} < \bar{s}_1 = \bar{s}_2 < s_1 = s_2 \ .$$

Therefore

$$\begin{aligned} \|(R'_\varrho(f,\mu) - R'_\varrho(g,\nu))(h_1,h_2)\| &\le c\|(f,\mu)-(g,\nu)\|_{H_0^{\bar{s}_1} \times H_0^{\bar{s}_2}} \|(h_1,h_2)\|_{H_0^{\bar{s}_1} \times H_0^{\bar{s}_2}} \\ &\le c\|(f,\mu)-(g,\nu)\|_{H_0^{\bar{s}_1} \times H_0^{\bar{s}_2}} \|(h_1,h_2)\|_{H_0^{s_1} \times H_0^{s_2}} \end{aligned}$$

and we have again

$$\|R'_\varrho(f,\mu) - R'_\varrho(g,\nu)\|_{H_0^{s_1} \times H_0^{s_2} \to L^2(S^1 \times [-\varrho,\varrho])} \le c\|(f,\mu)-(g,\nu)\|_{H_0^{\bar{s}_1} \times H_0^{\bar{s}_2}} \ . \tag{58}$$

The embedding from $H_0^{s_1} \times H_0^{s_2}$ to $H_0^{\bar{s}_1} \times H_0^{\bar{s}_2}$ is compact, and therefore weak convergence in $H_0^{s_1} \times H_0^{s_2}$ induces norm convergence in $H_0^{\bar{s}_1} \times H_0^{\bar{s}_2}$. We conclude for a sequence $(f_n,\mu_n) \rightharpoonup (f,\mu)$ for $n \to \infty$ in $D_{s_1,s_2,C} \subset H_0^{s_1} \times H_0^{s_2}$ that

$$\|R'_\varrho(f_n,\mu_n) - R'_\varrho(f,\mu)\|_{H_0^{s_1} \times H_0^{s_2} \to L^2(S^1 \times [-\varrho,\varrho])} \to 0 \text{ for } n \to \infty$$

holds, and consequently (27).

□

3.4 Conclusions

We have shown that Morozov's discrepancy principle for Tikhonov regularization applies to a wide class of problems. The existence of a regularization parameter α with (13) can be guaranteed under mild restrictions. If in addition (27) is assumed, then there exists for every sequence $\alpha_k \to \alpha$ a subsequence α_{k_n} with $x^\delta_{\alpha_{k_n}} \to x^\delta_\alpha$. In the above examples was demonstrated that (27) can often be concluded from the Lipschitz continuouity of the Frechét derivative of the nonlinear operator. We have shown that our conditions are easy to handle and apply even when (12) fails.

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