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Time-Harmonic Acoustic Wave Scattering in an Ocean with Depth-Dependent Sound Speed

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Abstract

Time-harmonic acoustic wave propagation in an inhomogeneous ocean with depth-dependent sound speed can be modeled by the Helmholtz equation in an infinite, two- or three-dimensional waveguide of finite height. Using variational theory in Sobolev spaces we prove well-posedness of the corresponding scattering problem from a bounded inhomogeneity inside such an ocean. To this end, we introduce an exterior Dirichlet-to-Neumann operator for depth-dependent sound speed and prove boundedness, coercivity, and holomorphic dependence of this operator in function spaces adapted to our weak solution theory. Analytic Fredholm theory then yields existence and uniqueness of solution for the scattering problem for all but a countable sequence of frequencies.

1 Introduction

Propagation of sound waves inside an ocean is an active research area in applied mathematics and engineering at least since the mid-20th century for its crucial importance for techniques like SONAR or for oil exploration (see, e.g., the introduction of [BGWX04] or [Buc92]). After the millenium change, precise models for sound propagation became even more important due to the observation that man-made ocean noise pollution endangers marine mammals and legal thresholds for emitted sound energies were set up. Checking these thresholds, e.g., for acoustic pulses produced by an air guns, requires sufficiently accurate models yielding quantitatively exact simulations of sonic intensities. One approach satisfying this requirement is to model scattering of time-harmonic acoustic waves in the ocean using the Helmholtz equation and to discretize this equation using established approximation technique as, e.g., finite elements or boundary elements.

It is well-known that a sound knowledge on variation theory of weak solutions in Sobolev spaces, in particular the existence of Gårding inequalities, is crucial for proving convergence of such numerical approximations, see [SS11]. However, to the best of our knowledge, weak solution theory for ocean scattering problems has up to now merely been set up for the case of a constant sound speed, restricting the applicability of this approach to shallow seas. For this reason, it is our aim in this paper is to provide rigorous theory for weak solutions via a variational approach for wave scattering in a flat ocean with variable, depth-dependent refractive index. The crucial and non-trivial difficulty compared to known results for constant sound speed, see [AGL08, AGL11], is that the eigenmodes of the ocean are not known explicitly for such a setting. In consequence, we exploit on the one hand via estimates for these modes and their eigenvalues and, on the other hand, obtain holomorphic dependence of the eigenvalues on the frequency from abstract perturbation theory.

Let us now introduce the model of sound propagation in the ocean investigated in the result of the paper. The domain of interest is a waveguide $\Omega = \mathbb{R}^m \times (0, H)$, $m = 2, 3$, where $H > 0$ is the constant depth. For points $x \in \Omega$, we write

$$x = (\tilde{x}, x_m)^\top \qquad \text{with } \tilde{x} = x_1 \text{ for } m = 2 \text{ and } \tilde{x} = (x_1, x_2)^\top \text{ for } m = 3.$$

If we denote by $\omega > 0$ the angular frequency and by $c : (0, H) \to \mathbb{R}$ the speed of sound depending on the depth of the ocean, then the propagation of time-harmonic sound waves with time-dependence $\exp(i\omega t)$ and small amplitude inside an inhomogeneous ocean is modeled by the Helmholtz equation

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)} u(x) = 0 \qquad \text{for } x \in \Omega. \tag{1}$$

In this setting, a local perturbation inside the inhomogeneous waveguide Ω is modeled by a refractive index $n^2 : \Omega \to \mathbb{C}$ such that the support of the contrast $q = n^2 - 1$ is a bounded set $D \subset \overline{\Omega}$, i.e., $\operatorname{supp}(q) = \overline{D}$, such that sound waves in the perturbed ocean satisfy

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)} (1 + q(x)) u(x) = 0 \qquad \text{for } x \in \Omega. \tag{2}$$

We assume in the following that the background sound speed $c \in L^\infty(0, H)$ satisfies

$$0 < c_- \leq c(x_m) \leq c_+ \qquad \text{for almost all } x_m \in (0, H). \tag{3}$$

This implies that

$$0 < \frac{\omega}{c_+} \leq \frac{\omega}{c(x_m)} \leq \frac{\omega}{c_-} \qquad \text{for almost all } x_m \in (0, H). \tag{4}$$

We model the free surface of the ocean by a sound-soft boundary and the seabed of the ocean by a sound-hard boundary,

$$u = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^3 : x_m = 0\} \qquad \text{and} \qquad \frac{\partial u}{\partial x_m} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^3 : x_m = H\}, \tag{5}$$

respectively. This setting yields a sufficiently accurate to model acoustic waves with small amplitude in a sea with negligible seabed variation. More flexible boundary models for, e.g., the ocean-seabed interface exist, see, e.g., [GL97], but for simplicity we restrict ourselves to the simpler condition Neumann condition from (5) describing a perfectly reflecting bottom.

When an incident sound field u^i satisfies the unperturbed Helmholtz equation (1) subject to the waveguide boundary conditions (5) then the inhomogeneous medium described by q causes a scattered field u^s such that the total field $u = u^i + u^s$ solves the perturbed Helmholtz equation (2) with contrast q, subject to $u = 0$ on Γ_0 and $\partial u / \partial \nu = 0$ on Γ_H. On interfaces where q jumps we prescribe that both the trace and the normal derivative of u are continuous across the interface. To ensure uniqueness of solution we further need to impose a radiation condition on u. This condition will be constructed with the help of a modal analysis in Section 3 below. Note that we seek for weak solutions to the scattering problem, i.e., for a function u that is locally in H^1 and satisfies

$$\int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_3)} (1 + q(x)) u \overline{v} \right) dx = 0 \qquad \text{for all } v \in C_0^\infty(\Omega).$$

The subsequent sections are organized as follows: Seeking solutions to (1) by separation of variables, a Liouville eigenvalue problem turns up that we investigate in Section 2. After showing holomorphic dependence of the eigenvalues on the frequency, we use these eigenvalues in Section 3 rigorously set up the scattering problem we investigate, and in Section 4 to prove spectral characterizations of Sobolev-type function spaces. Those are exploited in Sections 5 and 6 for analyzing the exterior Dirichlet-to-Neumann operator for the waveguide scattering problem. Finally, Section 7 contains and proves the main existence and uniqueness result of the paper via a Gårding inequality and analytic Fredholm theory.

2 A Liouville Eigenvalue Problem

We start by seeking solutions u to the Helmholtz equation (1) with boundary conditions (5) by separation of variables. Separating the horizontal variable $\tilde{x}$ and the vertical variable x_m in the form $u(\tilde{x}, x_m) = w(\tilde{x})\phi(x_m)$ one finds that w and ϕ need to solve the differential equations

$$\frac{\Delta_{\tilde{x}} w(\tilde{x})}{w(\tilde{x})} = -\frac{\phi''(x_m)}{\phi(x_m)} - \frac{\omega^2}{c^2(x_m)} =: \lambda^2 \qquad \text{in } \Omega \tag{6}$$

for some constant $\lambda \in \mathbb{C}$. (Here, $\Delta_{\tilde{x}}$ is the $m-1$-dimensional Laplacian in the variables $\tilde{x}$.) We first investigate the eigenvalue problem

$$\phi''(x_m) + \left[\frac{\omega^2}{c^2(x_m)} + \lambda^2 \right] \phi(x_m) = 0 \qquad \text{in } (0, H) \tag{7}$$

with boundary conditions $\phi_j(0) = 0$ and $\phi'(H) = 0$ corresponding to the boundary conditions (5) for the solution u to the Helmholtz equation (1). To this end, we consider weak solutions to this eigenvalue problem in the Sobolev space

$$H^1_W(0, H) := \{\psi \in H^1(0, H) : \psi(0) = 0\}.$$

The latter space is well-defined by the well-known continuous embedding of $H^1(0, H)$ in the Hölder space $C^{0,1/2}(0, H)$. Multiplying (7) with a test function $\psi \in H^1_W([0, H])$, formally integrating by parts and plugging in the boundary conditions for ϕ shows that the weak formulation of this eigenvalue problem is to find an eigenvalue $\lambda^2 \in \mathbb{C}$ and a corresponding eigenfunction $\phi \in H^1_W([0, H])$ such that

$$a(\phi, \varphi) := \int_0^H \left(\phi' \, \overline{\psi}' - \frac{\omega^2}{c^2(x_m)} \phi \, \overline{\psi} \right) dx_m \stackrel{!}{=} \lambda^2 \int_0^H \phi \, \overline{\psi} \, dx_m \quad \text{for all } \psi \in H^1_W([0, H]). \tag{8}$$

The sesquilinear form a on the left is obviously bounded in $H^1_W([0, H])$ and Poincaré's inequality together with the compact embedding of $H^1_W([0, H])$ in $L^2(0, H)$ shows that a is coercive in $H^1_W([0, H])$ up to a compact perturbation. Since ω/c^2 is real-valued, a is moreover symmetric, i.e., $a(\varphi, \psi) = a(\psi, \varphi)$ for all $\varphi, \psi \in H^1_W([0, H])$. Thus, the eigenvalue theory for self-adjoint and coercive variational problems in, e.g., [McL00, Theorem 2.7], shows that there exists a sequence of eigenvalues $\{\lambda_j^2\}_{j \in \mathbb{N}} \subset \mathbb{R}$ such that $\lambda_j^2 \to +\infty$ as $j \to \infty$ and associated eigenfunctions $\phi_j \in H^1_W([0, H])$ that are orthonormal in $L^2(0, H)$. We order the eigenvalues $\lambda_j^2 \in \mathbb{R}$ in increasing order, i.e., $-\infty < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \dots$ and define their square roots by

$$\lambda_j = \begin{cases} \sqrt{\lambda_j^2} & \text{if } \lambda_j^2 \geq 0 \text{ and} \\ -i\sqrt{|\lambda_j^2|} & \text{if } \lambda_j^2 < 0, \end{cases} \tag{9}$$

and extend the square root function from to positive real axis to a holomorphic function in the slit complex plane with branch cut along the positive imaginary axis. The definition of a weak derivative of a one-dimensional function in connection with the variational equation

$$\int_0^H \left[\frac{\omega^2}{c^2(x_m)} + \lambda_j^2 \right] \phi_j \, \overline{\psi} \, dx_m = \int_0^H \phi_j' \, \overline{\psi}' \, dx_m \qquad \text{for all } \psi \in H^1_W([0, H]).$$

shows that ϕ_j' belongs to $H^1([0, H])$. We conclude that $\phi \in H^2([0, H])$ and that the eigenvalue equation

$$\phi_j''(x_m) + \left[\frac{\omega^2}{c^2(x_m)} + \lambda_j^2 \right] \phi_j(x_m) = 0$$

holds in $L^2(0, H)$. Since $H^2([0, H])$ embeds continuously into the Hölder space $C^{1,1/2}([0, H])$ it holds that $\phi_j \in C^{1,1/2}([0, H])$ satisfies the boundary conditions $\phi_j(0) = 0$ and $\phi'(H) = 0$ hold in the classical sense of a point evaluation.

Remark 2.1. *It is well-known that the eigenpairs* $(\lambda_j, \phi_j)_{j\in\mathbb{N}}$ *for constant background sound speed* $c_\pm$ *are given by*

$$\lambda_j^2 = \Big(\frac{\pi}{2H}(2j-1)\Big)^2 - \frac{\omega^2}{c_\pm^2} \quad \text{and} \quad \phi_j(x_m) = \sin\Big(\frac{\pi}{2H}(2j-1)x_m\Big), \quad x_m \in [0, H].$$

Lemma 2.2. *(a) For* $j \in \mathbb{N}$ *it holds that*

$$\Big(\frac{\pi}{2H}(2j-1)\Big)^2 - \frac{\omega^2}{c_-^2} \le \lambda_j^2 \le \Big(\frac{\pi}{2H}(2j-1)\Big)^2 - \frac{\omega^2}{c_+^2}. \tag{10}$$

In consequence, the absolute value $|\lambda_j^2|$ *grows quadratically as* $j \to \infty$.
(b) There are constants $0 < c < C$ *such that* $cj \le \|\phi_j'\|_{L^2(0,H)} \le Cj$ *and* $\|\phi_j'\|_{L^2(0,H)} \le C(1 + |\lambda_j|^2)^{1/2}$ *for all* $j \in \mathbb{N}$.
(c) There is C *independent of* $j \in \mathbb{N}$ *such that* $|\phi_j(x_m)| \le C$ *for* $0 \le x_m \le H$.

Proof. (a) The min-max theorem implies for all $j \in \mathbb{N}$ that

$$\begin{aligned}
\lambda_j^2 &= \min_{V_j \subset H_W^1([0,H]), \dim(V_j)=j} \ \max_{\phi_j \in V_j, \|\phi_j\|=1} a(\phi_j, \phi_j) \\
&= \min_{V_j \subset H_W^1([0,H]), \dim(V_j)=j} \ \max_{\phi_j \in V_j, \|\phi_j\|=1} \int_0^H \big(|\phi_j'|^2 - \frac{\omega^2}{c(x_m)^2}|\phi_j|^2\big) dx_m \\
&\lesseqgtr \min_{V_j \subset H_W^1([0,H]), \dim(V_j)=j} \ \max_{\phi_j \in V_j, \|\phi_j\|=1} \int_0^H \big(|\phi_j'|^2 - \frac{\omega^2}{c_\pm^2}|\phi_j^+|^2\big) dx_m = \Big(\frac{\pi}{2H}(2j-1)\Big)^2 - \frac{\omega^2}{c_\pm^2}.
\end{aligned}$$

Since $(\pi(2j-1)/(2H))^2$ grows quadratically in j as $j \to \infty$ there exists $C > 0$ such that $|\lambda_j^2| \le Cj$ for all $j \in \mathbb{N}$.

(b) By a partial integration, the boundary conditions $\phi_j(0) = 0$ and $\phi_j'(H) = 0$ yield

$$\int_0^H |\phi_j'|^2 dx_m = -\int_0^H \phi_j'' \, \overline{\phi_j} dx_m + \Big[\phi_j \, \phi_j'\Big]_0^H = \int_0^H \Big[\frac{\omega^2}{c^2(x_m)} + \lambda_j^2\Big] |\phi_j|^2 dx_m. \tag{11}$$

Since $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis in $L^2([0,H])$, part (a) implies that

$$\frac{\pi^2(2j-1)^2}{4H^2} + \omega^2 \frac{c_-^2 - c_+^2}{c_+^2 c_-^2} \le \frac{\omega^2}{c_+^2} + \lambda_j^2 \le \|\phi_j'\|_{L^2(0,H)}^2 \le \frac{\omega^2}{c_-^2} + \lambda_j^2 \le \frac{\pi^2(2j-1)^2}{4H^2} + \omega^2 \frac{c_+^2 - c_-^2}{c_+^2 c_-^2}.$$

Thus, choosing $c = \pi^2/(2H)^2$ and $C = \pi^2/H^2 + \omega^2(1/c_-^2 - 1/c_+^2)$ implies that $0 < cj \le \|\phi_j'\|_{L^2(0,H)} \le Cj$ for all $j \in \mathbb{N}$. The second estimate follows from part (a).

(c) To show the uniform boundedness of ϕ_j we recall that (λ_j, ϕ_j) satisfies

$$\phi_j''(x_m) + \lambda_j^2 \phi_j(x_m) = \underbrace{-\frac{\omega^2}{c^2(x_m)} \phi_j(x_m)}_{=: f(x_m)} \tag{12}$$

with boundary conditions $\phi_j(0) = 0$ and $\phi_j'(H) = 0$. Interpreting the latter eigenvalue problem as a boundary value problem in $[0, H]$ we choose the ansatz $\phi_j(x_m) = \alpha(x_m)\exp(i\lambda_j x_m)$ for its solution and note that

$$\begin{aligned}
\phi_j'(x_m) &= \alpha'(x_m)\exp(i\lambda_j x_m) + i\lambda_j \alpha(x_m)\exp(i\lambda_j x_m), \\
\phi_j''(x_m) &= \alpha''(x_m)\exp(i\lambda_j x_m) + 2i\lambda_j \alpha'(x_m)\exp(i\lambda_j x_m) - \lambda_j^2 \phi_j(x_m).
\end{aligned}$$

Next, we insert $\phi_j''(x_m)$ into (12) to get that $\alpha''(x_m) + 2i\lambda_j\alpha'(x_m) = \exp(-i\lambda_j x_m) f(x_m)$, i.e.,

$$(\alpha'(x_m)\exp(2i\lambda_j x_m))' = f(x_m)\exp(i\lambda_j x_m).$$

Twice integrating this equation shows that

$$\alpha(x_m) = \alpha(0) + \alpha'(H)x_m - \int_0^{x_m} \exp(-i\lambda_j s) \int_s^H f(t)\exp(i\lambda_j t)\, dt\, ds. \tag{13}$$

Since the Dirichlet boundary condition shows that $\alpha(0) = 0$, we plug $x_m = H$ into the last equation and obtain

$$\alpha(H) = \alpha'(H)H - \int_0^H \exp(-i\lambda_j s) \int_s^H f(t)\exp(i\lambda_j t)\, dt\, ds =: \alpha'(H)H + C. \tag{14}$$

Since $\alpha(x_m) = \phi_j(x_m)\exp(-i\lambda_j x_m)$ it holds that

$$\alpha'(x_m) = \phi_j'(x_m)\exp(-i\lambda_j x_m) - i\lambda_j \exp(-i\lambda_j x_m)\phi_j(x_m).$$

Choosing $x_m = H$ in the last equation shows that

$$\alpha'(H) = -i\lambda_j \underbrace{\exp(-i\lambda_j H)\phi_j(H)}_{=\alpha(H)}. \tag{15}$$

Consequently, equation (14) and (15) imply that

$$\begin{pmatrix} 1 & -H \\ i\lambda_j & 1 \end{pmatrix} \begin{pmatrix} \alpha(H) \\ \alpha'(H) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix}$$

that is, $\alpha'(h) = -i\lambda_j C/(1 + i\lambda_j H)$ and, due to (13),

$$\alpha(x_m) = C\frac{-i\lambda_j}{1 + i\lambda_j H} x_m - \int_0^{x_m} \exp(-i\lambda_j s) \int_s^H f(t) \exp(i\lambda_j t)\, dt\, ds.$$

Plugging the last equation into the ansatz $\phi_j(x_m) = \alpha(x_m) \exp(i\lambda_j x_m)$ and then applying the Cauchy-Schwartz inequality finishes the proof. $\square$

The eigenvalues λ_j^2 obviously depend on the frequency $\omega > 0$. Writing $\lambda_j^2 = \lambda_j^2(\omega)$ we show next that $\omega \mapsto \lambda_j^2(\omega)$ can be extended as a holomorphic function into a complex open neighborhood of $\mathbb{R}_{>0}$ in $\mathbb{C}$.

Lemma 2.3. *For all $\omega_* > 0$ there exists an open neighborhood $U(\omega_*) \subset \mathbb{C}$ and and index functions $\ell_j : U(\omega_*) \to \mathbb{N}$ that satisfy $\cup_{j\in\mathbb{N}} \ell_j(\omega) = \mathbb{N}$ and $\ell_j(\omega) \neq \ell_j'(\omega)$ for $j \neq j' \in \mathbb{N}$ and all $\omega \in U$, such that the eigenvalue curves $\omega \mapsto \lambda^2_{\ell_j(\omega)}(\omega)$ are real-analytic functions in $U(\omega_*) \cap \mathbb{R}$ and extend to holomorphic functions in $U(\omega_*)$ for all $j \in \mathbb{N}$.*

Proof. We exploit results on holomorphic families of operators from [Kat95, Chapter VII, §2 and §4]. Choose some $\omega_* > 0$. The differential operators $L(\omega)\, u = u'' + ((\omega_*)^2/c^2)u$ on $(0, H)$ with boundary conditions $u(0) = 0$ and $u'(H) = 0$ yield a selfadjoint holomorphic family of type (A) since $u \mapsto (\omega_*^2/c^2)u$ is bounded on $L^2(0, H)$, $\omega_* \mapsto (\omega_*^2/c^2)u$ is holomorphic in $\omega_* \in \mathbb{C}$, and the domain $\{u \in H^2(0, H),\, v(0) = 0\}$ of $L(\Omega)$ is independent of $\omega_* \in \mathbb{C}$, compare [Kat95, Ch. VII, §1.1, §2.1, Th. 2.6]. These differential operators also form of a holomorphic family of type (B) since the associated sesquilinear form a from (8) is bounded.

From [Kat95, Ch. VII, §3.1, Example 4.23] it follows that for each eigenvalue $\lambda_j^2(\omega_*)$, $j \in \mathbb{N}$, with multiplicity one that there is a complex neighborhood U_j of ω_* such that $\omega \mapsto \lambda_j^2(\omega)$ can be extended from $U_j \cap \mathbb{R}$ as a holomorphic function of ω into U_j. If $\lambda_j^2(\omega_*)$ is a multiple eigenvalue, then the function $\omega \mapsto \lambda_j^2(\omega)$ is in general not differentiable at ω^*, such that the eigenvalue index needs to be re-ordered to obtain smooth eigenvalue curves, compare [Kat95, Ch. VII, §3.1, Ch. 2, Th. 6.1]. Indeed, the latter reference shows that if $\lambda_j^2(\omega_*)$ is a multiple eigenvalue then it has finite multiplicity and there exists a complex neighborhood U_j of ω_* and an index function $\ell_j : U_j \cap \mathbb{R} \to \mathbb{N}$ such that $\omega \mapsto \lambda^2_{\ell_j(\omega)}(\omega)$ can be extended holomorphically from $U_j \cap \mathbb{R}$ into U_j. Thus, while the curves $\omega \mapsto \lambda_j^2(\omega)$ are merely piecewise analytic for real $\omega > 0$ and the corresponding eigenvalue sheets are piecewise holomorphic in a complex neighborhood of $\mathbb{R}_{>0}$), analyticity can be obtained by re-ordering indices via the index functions ℓ_j.

It remains to show that the intersection of the neighborhoods U_j is non-empty. This is certainly true for any finite union $\cup_{j=1}^N U_j$ with $N \in \mathbb{N}$. Moreover, the eigenvalue estimates (10) imply that the distance d_j of $\lambda_j^2(\omega_*)$ to the rest $\{\lambda_\ell^2(\omega_*),\, \lambda_\ell^2(\omega_*) \neq \lambda_j^2(\omega_*)\}$ of the spectrum of $L(\omega_*)$, i.e.,

$$d_j = \begin{cases} \lambda_j^2(\omega) - \lambda_{j-1}^2(\omega) & j \geq 2, \\ \lambda_2^2(\omega) - \lambda_1^2(\omega) & j = 1, \end{cases}$$

is bounded from below by 1 whenever $j > j_* \in \mathbb{N}$, with

$$j_* := \left\lceil \frac{\omega_*^2 H^2}{2\pi^2} \left(\frac{1}{c_-^2} - \frac{1}{c_+^2} \right) + \frac{H^2}{2\pi^2} \right\rceil .$$

Thus, Theorem 4.8 in [Kat95, Ch. VII], compare also (4.45) in the same chapter, implies that for all $j > j_*$ the holomorphic extension of $\lambda_j^2(\omega_*)$ has a convergence radius of at least $(1 + \|q\|_{L^\infty(0,H)})^{-1}$. (Set $\varepsilon = 2$ and $a = 1$, $b = 0$, and $c = \|q\|_{L^\infty(0,H)}$ in (4.45).) In particular, all eigenvalues $\lambda_j^2(\omega_*)$ extend to a holomorphic functions in $U(\omega_*) := \cup_{j=1}^{j_*} U_j \cup B(\omega_*, 1)$. □

Theorem 2.4. *There exists an open complex neighborhood U of $\mathbb{R}_{>0}$ and index functions $\ell_j : U \to \mathbb{N}$ such that the eigenvalue curves $\lambda_{\ell_j(\omega)}^2(\omega)$ are real-analytic curves that extend to holomorphic functions in U for all $j \in \mathbb{N}$. For each compact subset W of U, the set $K_0 = \{\omega \in W$, there is $j \in \mathbb{N}$ such that $\lambda_j^2(\omega) = 0\}$ is finite.*

Proof. We cover the positive real half axis $[0, \infty)$ with the neighborhoods $U(\omega)$ of $\omega > 0$ constructed in Lemma 2.3. For each compact interval $[0, \ell]$ for $\ell \in \mathbb{N}$ there exists a finite sub cover, which allows to continue the real eigenvalue functions $\omega \mapsto \lambda_{\ell_j(\omega)}^2(\omega)$ into a complex neighborhood of $[0, n]$ for all $j \in \mathbb{N}$ and $n \in \mathbb{N}$. Finally, Theorems 1.9 and 1.10 in [Kat95, Ch. VII, §1.3] state that on compact subsets W of U either for each number $\omega \in W$ there is $j = j(\omega) \in \mathbb{N}$ such that $\lambda_j(\omega)^2 = 0$ or that the number of such ω is finite. Since the first alternative obviously does not hold, the above-introduced set K_0 is finite. □

3 The Scattering Problem

In this section we rigorously set up the mathematical formulation of acoustic scattering in the above-introduced ocean model, based on the eigenpairs $(\lambda_j, \phi_j)_{j \in \mathbb{N}}$.

We first go back to the construction of solutions to the Helmholtz equation (1) by separation of variables and note that the series

$$u(\tilde{x}, x_m) = \sum_{j \in \mathbb{N}} c(j)\, w_j(\tilde{x}) \phi_j(x_m) \qquad \text{with coefficients } c(j) \in \mathbb{C} \tag{16}$$

is a formal solution to the Helmholtz equation whenever $w_j : \mathbb{R}^{m-1} \to \mathbb{C}$ solves

$$\Delta_{\tilde{x}} w_j - \lambda_j^2 w_j = 0 \qquad \text{in } \mathbb{R}^{m-1}. \tag{17}$$

Thus, the eigenfunctions ϕ_j give for instance rise to plane wave-solutions $u_j(x;\theta) = \exp(i\lambda_j\,\theta\cdot\tilde{x})\phi_j(x_m)$ of the Helmholtz equation in Ω with direction $\theta \in \mathbb{R}^{m-1}$ such that $|\theta|_2 = 1$. These so-called waveguide modes satisfy the waveguide boundary conditions at $\Gamma_{0,H}$ by construction of ϕ_j. They are called propagating whenever $\lambda_j \in i\mathbb{R}$ (i.e., $\lambda_j^2 < 0$) and evanescent whenever $\lambda_j \in \mathbb{R}$ (i.e., $\lambda_j^2 > 0$). The number of such propagating modes is (up to rotation or reflection) determined by the largest integer $J = J(\omega, c, H)$ such that $\lambda_J^2 < 0$. Whenever $\lambda_j^2 = 0$, the mode u_j is obviously constant in $\tilde{x}$ – this somewhat exceptional will be investigated later on.

If one aims to find physically meaningful scattered fields via the series representation (16) one additionally needs to prescribe radiation conditions for the functions w_j: If $i\lambda_j \in \mathbb{R}_{>0}$ is purely imaginary then Sommerfeld's radiation condition determines solutions that are outwards radiating in any horizontal direction; if $\lambda_j \in \mathbb{R}_{>0}$ is positive then we prescribe that w_j must be a bounded solution to (17). Obviously, whenever $\lambda_j = \lambda_j^2 = 0$ for some $j \in \mathbb{N}$ neither of two classifications applies (the corresponding mode does not depend on $\tilde{x}$); in this case, we call the frequency $\omega > 0$ an exceptional frequency. As in corresponding studies of scattering in waveguides with constant sound speed, see, e.g., [AGL08], we exclude this case from now on. (We will show later that such exceptional frequencies form an at most countable set without finite accumulation point.)

Assumption 3.1. *In the sequel we assume that the frequency $\omega > 0$ is chosen such that $\lambda_j^2 \neq 0$ for all $j \in \mathbb{N}$.*

Under this assumption we call a solution u to the Helmholtz equation radiating whenever for some $\rho_0 > 0 > 0$ and all $x \in \Omega$ such $|\tilde{x}| > \rho_0$ one can represent u in the form (16) with solutions w_j to $(\Delta_{\tilde{x}} - \lambda_j^2)w_j = 0$ in $\{|\tilde{x}| > \rho_0\}$ such that

$$\text{if } i\lambda_j \in \mathbb{R}_{>0} \text{ then } \lim_{|\tilde{x}|\to\infty} \sqrt{\tilde{x}}\Big(\frac{\partial w_j}{\partial|\tilde{x}|} + \lambda_j w_j\Big) = 0 \text{ uniformly in } \frac{\tilde{x}}{|\tilde{x}|}, \tag{18}$$

whereas

$$\text{if } \lambda_j \in \mathbb{R}_{>0} \text{ then } w_j(\tilde{x}) \text{ is uniformly bounded for } |\tilde{x}| > \rho_0. \tag{19}$$

Remark 3.2. *The sign in front of λ_j in the radiation condition* (18) *for the propagating modes is indeed correct: As $i\lambda_j \in \mathbb{R}_{>0}$, i.e., $\lambda_j \in i\mathbb{R}_{<0}$,* (18) *implies that $(\partial w_j/\partial|\tilde{x}|) - i|\lambda_j|\, w_j \to 0$ as $|\tilde{x}| \to \infty$. This difference from the usual convention in scattering theory is due to the choice of the sign of λ_j^2 in* (6) *that respects the standard choice of Sturm-Liouville theory.*

Recall from the introduction that the contrast function $q : \Omega \to \mathbb{C}$ is supported in the scattering object D supposed to be a bounded Lipschitz domain included in $\overline{\Omega}$. It is moreover physically reasonable to assume that $\mathrm{Im}(q) \geq 0$, i.e., we allow for energy absorption inside the penetrable scatterer D. When a source emits an incident sound field u^i that satisfies the unperturbed Helmholtz equation subject to the waveguide boundary conditions,

$$\Delta u^i(x) + \frac{\omega^2}{c^2(x_m)} u^i(x) = 0 \text{ for } x \in \Omega, \quad u^i(x) = 0 \text{ for } x \in \Gamma_0 \quad \text{and} \quad \frac{\partial u^i}{\partial x_m}(x) = 0 \text{ for } x \in \Gamma_H, \tag{20}$$

then the inhomogeneous medium described by q causes a scattered field u^s such that the total field $u = u^i + u^s$ solves the perturbed Helmholtz equation with contrast q, i.e.,

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)}(1 + q(x))u(x) = 0 \qquad \text{for } x \in \Omega, \tag{21}$$

subject to $u = 0$ on Γ_0 and $\partial u/\partial \nu = 0$ on Γ_H and, additionally, the scattered field u^s is radiating, i.e., possesses a representation of the form (16) that satisfies (18–19). On interfaces where q jumps we prescribe that both the trace and the normal derivative of u are continuous across the interface.

Since we are interested in existence theory of weak solutions to this scattering problem, we define for $\rho > 0$ domains $\Omega_\rho = \{x \in \Omega, |\tilde{x}| < \rho\}$ and for arbitrary Lipschitz domains $U \subset \overline{\Omega}$ the Sobolev space

$$H^1_W(U) = \left\{ v \in H^1(U),\ v|_{U \cap \{x_m = 0\}} = 0 \right\}.$$

This space is well-defined due to the well-known trace theorem in H^1. For $l \in \mathbb{N}$ we further set

$$H^l_{W,\mathrm{loc}}(\Omega) = \left\{ v : \Omega \to \mathbb{C},\ v|_{\Omega_\rho} \in H^1_W(\Omega_\rho) \cap H^l(\Omega_\rho) \text{ for all } \rho > 0 \right\}.$$

Now we can rigorously formulate the above-introduced scattering problem: Given $c \in L^\infty(0, H)$ such that $0 < c_- \leq c \leq c_+$, $q \in L^\infty(\Omega)$ such that $\mathrm{Im}(q) \geq 0$ and $\mathrm{supp}(q) \subset \Omega_\rho$, and $u^i \in H^2_{W,\mathrm{loc}}(\Omega)$ that satisfies the Helmholtz equation in (20) in $L^2_{\mathrm{loc}}(\Omega)$ and the waveguide boundary conditions in (20) in the trace sense, we seek for $u \in H^1_{W,\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)}(1 + q) u \overline{v} \right) dx = 0 \qquad \text{for all compactly supported } v \in H^1_W(\Omega), \tag{22}$$

and, additionally, for some $\rho_0 > 0$ it holds that

$$u^s(x) = u(x) - u^i(x) = \sum_{j \in \mathbb{N}} c(j)\, w_j(\tilde{x}) \phi_j(x_m) \qquad \text{for all } |\tilde{x}| > \rho_0. \tag{23}$$

The latter series is required to converge in $H^1(\Omega_\rho \setminus \Omega_{\rho_0})$ for all $\rho > \rho_0$ and the solutions $w_j \in C^\infty(|\tilde{x}| > \rho_0)$ to the Helmholtz equation $(\Delta_{\tilde{x}} - \lambda_j^2) w_j = 0$ in $|\tilde{x}| > \rho_0$ need to satisfy that

$$\begin{cases} \lim_{|\tilde{x}|\to\infty} \sqrt{\tilde{x}}\left(\frac{\partial w_j}{\partial|\tilde{x}|} + \lambda_j w_j\right) = 0 \text{ uniformly in } \frac{\tilde{x}}{|\tilde{x}|} & \text{if } i\lambda_j \in \mathbb{R}_{>0}, \\ w_j(\tilde{x}) \text{ is uniformly bounded for } |\tilde{x}| > \rho_0 & \text{if } \lambda_j \in \mathbb{R}_{>0}, \end{cases} \quad \text{for all } j \in \mathbb{N}. \tag{24}$$

Any solution to the Helmholtz equation outside Ω_ρ that satisfies (24) for all $j \in \mathbb{N}$ is in the sequel called a radiating solution.

Remark 3.3 (Radiating solutions are well-defined)**.** *Any solution v that solves the Helmholtz equation and the boundary conditions in* (20) *in $\Omega \setminus \overline{\Omega_{\rho_0}}$ that belongs to $H^1_{\mathrm{loc}}(\Omega \setminus \overline{\Omega_{\rho_0}})$ can be represented in series form as in* (23) *since the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}} \subset H^1_W(0,H)$ are a complete orthonormal system of $L^2(0,H)$. Thus, the above assumption on the scattered field u^s merely requires the conditions* (24) *to be satisfies. The series representation automatically follows from the fact that $u^s \in H^1_{\mathrm{loc}}(\Omega\setminus\overline{\Omega_{\rho_0}})$ satisfies the homogeneous Helmholtz equation and the waveguide boundary conditions. In particular, the radiation and boundedness conditions are well-defined for any such solution to the Helmholtz equation.*

4 Characterizations of Function Spaces

To analyze whether the scattering problem defined in the last section possesses a unique solution we will transform it into a variational problem on the bounded domain Ω_ρ for $\rho > 0$ chosen so large that $\mathrm{supp}(q) \subset \overline{\Omega_{\rho'}}$ for some $\rho' < \rho$. To this end, we define and analyze exterior Dirichlet-to-Neumann operators in the next section. In the remainder of this section, we introduce several technical tools and results for those operators; note that all results hold true even if Assumption 3.1 it not satisfied. First, we introduce the weighted inner product

$$\langle \phi, \psi \rangle_\diamond = \int_0^H \frac{\omega^2}{c^2(x_m)} \phi(x_m)\overline{\psi(x_m)}\, dx_m \qquad \text{on } L^2(0,H). \tag{25}$$

Obviously, the norm defined by the inner product $\langle \cdot, \cdot \rangle_\diamond$ is equivalent to the standard norm in $L^2(0,H)$, i.e., there is $C > 0$ such that $C^{-1}\langle \phi, \phi \rangle_\diamond \le \|\phi\|^2_{L^2(0,H)} \le C\langle \phi, \phi \rangle_\diamond$ for all $\phi \in L^2(0,H)$. Further, as for any separable Hilbert space there exists an orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}}$ of $(L^2(0,H), \langle \cdot, \cdot \rangle_\diamond)$, i.e., the ψ_j form a dense subset of $L^2(0,H)$ and satisfy

$$\int_0^H \frac{\omega^2}{c^2(x_m)} \psi_j(x_m)\overline{\psi_l(x_m)}\, dx_m = \delta_{j,l} \qquad j, l \in \mathbb{N}. \tag{26}$$

Second, we note that the standard theory on orthogonal bases in Hilbert spaces allows to expand a function $u \in L^2(\Omega_\rho)$ into its Fourier series with respect to the basis $\{\phi_j\}_{j\in\mathbb{N}}$,

$$u(x) = \sum_{j=1}^{\infty} \hat{u}(j,\tilde{x})\phi_j(x_m) \qquad \text{where } \hat{u}(j,\tilde{x}) = \int_0^H u(\tilde{x},x_m)\overline{\phi_j}(x_m)\,dx_m. \tag{27}$$

The latter series converges in $L^2(\Omega_\rho)$ and Parseval's identity states that

$$\|u\|^2_{L^2(\Omega_\rho)} = \sum_{j=1}^{\infty} \|\hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} \qquad \text{for } u \in L^2(\Omega_\rho).$$

Note that we use the notation $\hat{u}(j,\tilde{x})$ also for vector-valued functions $u \in L^2(\Omega_\rho)^l$ with l components.

Lemma 4.1. *(a) For $u \in H^1_W(\Omega_\rho)$ it holds that the coefficients $\hat{u}(j,\cdot)$ from* (27) *belong to $H^1(\{|\tilde{x}| < \rho\})$, and*

$$\|\nabla_{\tilde{x}} u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{Z}} \|\nabla_{\tilde{x}}\hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})}.$$

(b) If $u \in C^2(\overline{\Omega_\rho})$ then the series expansion (27) *converges absolutely and uniformly. Additionally, this expansion can be derived term by term with respect to x_m and the resulting series representation converges absolutely and uniformly,*

$$\frac{\partial u}{\partial x_m}(x) = \sum_{j=1}^{\infty} \hat{u}(j,\tilde{x})\phi_j'(x_m) \qquad \text{for } x \in C(\overline{\Omega_\rho}). \tag{28}$$

Proof. (a) The function $u = \sum_{j=1}^{\infty} \hat{u}(j,\tilde{x})\phi_j(x_m) \in L^2(\Omega_\rho)$ belongs to $H^1(\Omega_\rho)$ if and only if its first-order distributional derivatives all belong to $L^2(\Omega_\rho)$. Since $u \mapsto \partial u/\partial x_i$ is a continuous operation from $H^1(\Omega_\rho)$ into $L^2(\Omega_\rho)$, we can exchange this differential operator for $i = 1,\dots,m-1$ with the inner product of $L^2(0,H)$,

$$\int_0^H \frac{\partial u}{\partial x_i}(x)\overline{\phi_l}(x_m)\,dx_m = \frac{\partial}{\partial x_i}\int_0^H \sum_{j=1}^{\infty} \hat{u}(j,\tilde{x})\phi_j(x_m)\overline{\phi_l}(x_m)\,dx_m = \frac{\partial}{\partial x_i}\hat{u}(l,\tilde{x}). \tag{29}$$

The right-hand side is square-integrable, since the left-hand side can be estimated by

$$\begin{aligned}\left\|\int_0^H \frac{\partial u}{\partial x_i}(\tilde{x},x_m)\overline{\phi_l}(x_m)\,dx_m\right\|^2_{L^2(\{|\tilde{x}|<\rho\})} &\le \left\|\int_0^H \left|\frac{\partial u}{\partial x_i}(\tilde{x},x_m)\right|^2 dx_m\right\|_{L^2(\{|\tilde{x}|<\rho\})} \\ &\le \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\{|\tilde{x}|<\rho\},L^2(0,H))} = \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega_\rho)}\end{aligned}$$

Thus, all partial derivatives $\tilde{x} \mapsto \partial \hat{u}(l, \tilde{x})/\partial x_i$ for $i = 1, \ldots, m-1$ are square integrable, which implies that $\hat{u}(l, \tilde{x}) \in H^1(\{|\tilde{x}| < \rho\})$. By (29) and Parseval's identity,

$$\left\| \frac{\partial u}{\partial x_i} \right\|^2_{L^2(\Omega_\rho)} = \sum_{j=1}^{\infty} \left\| \frac{\partial}{\partial x_i} \hat{u}(l, \tilde{x}) \right\|^2_{L^2(\{|\tilde{x}|<\rho\})}, \qquad i = 1, \ldots, m-1.$$

(b) This follows from results on function expansions in terms of the Sturm-Liouville eigenfunctions $\{\phi_j\}_{j \in \mathbb{N}}$, compare, e.g., [LS60, Chapter 2, §4-§6]. □

In the next next lemma we write $\|u\|_1^2 \simeq \|u\|_2^2$ to indicate the equivalence of the two norms $\|\cdot\|_{1,2}$, i.e., the existence of $C > 0$ independent of u such that $C^{-1}\|u\|_1^2 \leq \|u\|_2^2 \leq C\|u\|_1^2$. Working in dimension $m = 2$ we further abbreviate the weak partial derivative $\partial \hat{u}(j, x_1)/\partial x_1$ by $\hat{u}'(j, x_1)$.

Lemma 4.2. *For $m = 2$ and $u \in H^1_W(\Omega_\rho)$ it holds that*

$$\|u\|^2_{H^1(\Omega_\rho)} \simeq \sum_{j=1}^{\infty} \int_{-\rho}^{\rho} \left[(1 + |\lambda_j|^2)|\hat{u}(j, x_1)|^2 + |\hat{u}'(j, x_1)|^2 \right] dx_1. \tag{30}$$

Proof. It is sufficient to show the claim for $u \in H^1_W(\Omega_\rho) \cap C^2(\overline{\Omega_\rho})$, since $H^1_W(\Omega_\rho) \cap C^2(\overline{\Omega_\rho})$ is a dense subset of $H^1_W(\Omega_\rho)$. For $u \in H^1_W(\Omega_\rho) \cap C^2(\overline{\Omega_\rho})$ Lemma 4.1 states that the series representation $(\partial u/\partial x_2)(x) = \sum_{j=1}^{\infty} \hat{u}(j, x_1)\phi_j'(x_2)$ of $\partial u/\partial x_2$ holds and additionally converges absolutely and uniformly in Ω_ρ.

Using the expression of the Fourier series in (27) we write $u_N(x) = \sum_{j=1}^{N} \hat{u}(j, x_1)\phi_j(x_2)$ and note that $u_N \to u$ as $N \to \infty$ in $H^1(\Omega_\rho)$ since u is twice differentiable, such that Lemma 4.1 applies. Clearly, $\|u_N\|^2_{L^2(\Omega_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{N} |\hat{u}(j, x_1)|^2 \, dx_1 \leq \|u\|^2_{L^2(\Omega_\rho)}$ and

$$\nabla u_N = \frac{\partial u_N}{\partial x_1} \boldsymbol{e}_1 + \frac{\partial u_N}{\partial x_2} \boldsymbol{e}_2 \qquad \text{where } \boldsymbol{e}_1 = (1, 0)^\top \text{ and } \boldsymbol{e}_2 = (0, 1). \tag{31}$$

Lemma 4.1 implies that $\|\partial u_N/\partial x_1\|^2_{L^2(\Omega_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{N} |\hat{u}'(j, x_1)|^2 \, dx_1$. As $u_N \to u$ in $H^1(\Omega_\rho)$ and since the norm is continuous, we can take the limit as $N \to \infty$ to obtain that

$$\|u\|^2_{L^2(\Omega_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{\infty} |\hat{u}(j, x_1)|^2 \, dx_1 \quad \text{and} \quad \left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(\Omega_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{\infty} |\hat{u}'(j, x_1)|^2 \, dx_1.$$

Moreover, a straightforward computation shows that

$$\left\| \frac{\partial u_N}{\partial x_2} \right\|^2_{L^2(\Omega_\rho)} = \int_{-\rho}^{\rho} \sum_{j,j'=1}^{N} \hat{u}(j', x_1)\overline{\hat{u}(j', x_1)} \, dx_1 \int_0^H \phi_j'(x_2)\overline{\phi_{j'}'}(x_2) \, dx_2.$$

The variational formulation of the eigenvalue problem (7) for (λ_j, ϕ_j) shows that

$$\Big\| \frac{\partial u_N}{\partial x_2} \Big\|^2_{L^2(\Omega_\rho)} = \int_{-\rho}^{\rho} \sum_{j,j'=1}^{N} \hat{u}(j,x_1)\overline{\hat{u}(j',x_1)} \int_0^H \Big(\frac{\omega^2}{c^2(x_2)} + \lambda_j^2 \Big) \phi_j(x_2)\phi_{j'}(x_2)\, dx_2 dx_1 \tag{32}$$
$$= \sum_{j,j'=1}^{N} \int_{-\rho}^{\rho} \hat{u}(j,x_1)\overline{\hat{u}(j',x_1)}\, dx_1 \, \Big[\lambda_j^2 \int_0^H \phi_j \overline{\phi_{j'}}\, dx_m + \langle \phi_j, \phi_{j'} \rangle_\diamond \Big]$$
$$= \sum_{j=1}^{N} \lambda_j^2 \int_{-\rho}^{\rho} |\hat{u}(j,x_1)|^2\, dx_1 \, + \sum_{j,j'=1}^{N} \int_{-\rho}^{\rho} \hat{u}(j,x_1)\overline{\hat{u}(j',x_1)}\, dx_1 \, \langle \phi_j, \phi_{j'} \rangle_\diamond$$

where we exploited the definition of the scalar product $\langle \cdot, \cdot \rangle_\diamond$ in (25). Recall that $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,H)$ for the standard inner product $(\phi, \varphi) \mapsto \int_0^H \phi\overline{\varphi}\, dx_m$. Since $\langle \cdot, \cdot \rangle_\diamond$ is equivalent to the standard inner product, Theorem 2.1 in Chapter VI of [GK69] implies that the infinite matrix $(\langle \phi_j, \phi_{j'} \rangle_\diamond)_{j,j=1}^\infty \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ defines a bounded and boundedly invertible operator

$$A: l^2 \to l^2, \qquad \{a_j\}_{j\in\mathbb{N}} \mapsto \Big\{ j' \mapsto \sum_{j=1}^{\infty} a_j \langle \phi_j, \phi_{j'} \rangle_\diamond \Big\}_{j'\in\mathbb{N}}. \tag{33}$$

Thus, the estimate $\sum_{j,j'=1}^N \hat{u}(j,x_1)\overline{\hat{u}(j',x_1)}\langle \phi_j, \phi_{j'} \rangle_\diamond \le \|A\|_{l^2\to l^2} \sum_{j=1}^N |\hat{u}(j,x_1)|^2$ holds due to the Cauchy-Schwartz inequality uniformly in N and implies that

$$\Big\| \frac{\partial u_N}{\partial x_2} \Big\|^2_{L^2(\Omega_\rho)} \le \sum_{j=1}^{N} (\|A\| + \lambda_j^2) \int_{-\rho}^{\rho} |\hat{u}(j,x_1)|^2\, dx_1 \le C \sum_{j=1}^{N} (1 + |\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}(j,x_1)|^2\, dx_1. \tag{34}$$

Since A is boundedly invertible, we moreover obtain the estimate

$$\sum_{j,j'=1}^{N} \int_{-\rho}^{\rho} \hat{u}(j,x_1)\overline{\hat{u}(j',x_1)}\, dx_1 \, \langle \phi_j, \phi_{j'} \rangle_\diamond \ge c \sum_{j=1}^{N} \int_{\rho}^{\rho} |\hat{u}(j,x_1)|^2\, dx_1$$

for some $c > 0$ independent of N. As above we exploit that $\partial u_N/\partial x_2 \to \partial u/\partial x_2$ in $L^2(\Omega_\rho)$ and continuity of the norm in $L^2(\Omega_\rho)$ to take the limit as $N \to \infty$ in (32) and to show that

$$\Big\| \frac{\partial u}{\partial x_2} \Big\|^2_{L^2(\Omega_\rho)} \ge \sum_{j=1}^{\infty} (c + \lambda_j^2) \int_{-\rho}^{\rho} |\hat{u}(j,x_1)|^2\, dx_1. \tag{35}$$

Now we show by a contradiction argument that there is $c_* > 0$ such that

$$\Big\| \frac{\partial u}{\partial x_2} \Big\|^2_{L^2(\Omega_\rho)} \geq c_* \sum_{j=1}^\infty (1 + |\lambda_j^2|) \int_{-\rho}^{\rho} |\hat{u}(j, x_1)|^2 \, dx_1 \qquad \text{for all } u \in H^1_W(\Omega_\rho). \tag{36}$$

Indeed, if the latter inequality does not hold uniformly for all $u \in H^1_W(\Omega_\rho)$ there is a sequence $\{u^{(l)}\}_{l\in\mathbb{N}} \subset H^1_W(\Omega_\rho)$ such that $\|\partial u^{(l)}/\partial x_2\|_{L^2(\Omega_\rho)} \to 0$ as $l \to \infty$ while $\sum_{j=1}^\infty (1 + |\lambda_j^2|)\|\hat{u}^{(l)}(j, \cdot)\|^2 = 1$ for all $l \in \mathbb{N}$.

To derive a contradiction, we show that $\|v\|_{L^2(\Omega_\rho)} \leq C\|(\partial v/\partial x_m)(\tilde{x}, \cdot)\|_{L^2(\Omega_\rho)}$ holds for all $v \in H^1_W(\Omega_\rho)$. To prove the latter estimate we note that for all $v \in H^1_W(\Omega_\rho) \cap C^2(\overline{\Omega_\rho})$ Poincaré's inequality in one dimension applied to $v(\tilde{x}, \cdot)$ shows that $\|v(\tilde{x}, \cdot)\|^2_{L^2(0,H)} \leq (H^2/2)\|(\partial v/\partial x_m)(\tilde{x}, \cdot)\|^2_{L^2(0,H)}$ and integration over $\tilde{x}$ implies that

$$\|v\|_{L^2(\Omega_\rho)} \leq H\|(\partial v/\partial x_m)(\tilde{x}, \cdot)\|_{L^2(\Omega_\rho)}. \tag{37}$$

Since $H^1_W(\Omega_\rho) \cap C^2(\overline{\Omega_\rho})$ is a dense subset of $H^1_W(\Omega_\rho)$ (37) holds for all $v \in H^1_W(\Omega_\rho)$. Recall from Section 2 that $J = J(\omega, c, H)$ denotes the number of propagating waveguide modes, that is, the number of eigenvalues λ_j^2 that are negative. Estimate (37) implies that

$$\sum_{j=1}^{J} (1+|\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}^{(l)}(j, x_1)|^2 \, dx_1 \leq C\|u^{(l)}\|^2_{L^2(\Omega_\rho)} \leq CH \Big\| \frac{\partial u^{(l)}}{\partial x_2} \Big\|^2_{L^2(\Omega_\rho)} \to 0 \quad \text{as } l \to \infty \tag{38}$$

for $C = \max_{1\leq j\leq J}(1 + |\lambda_j|^2)$. (Of course, the same argument holds for any finite truncation index; truncation at J is however sufficient for the following argument.) Thus, (35) directly yields

$$\sum_{j=J+1}^{\infty} (1+|\lambda_j^2|)\|\hat{u}^{(l)}(j, \cdot)\|^2 \leq \Big\| \frac{\partial u^{(l)}}{\partial x_2} \Big\|^2_{L^2(\Omega_\rho)} + \underbrace{\sum_{j=1}^{J} (|\lambda_j|^2 - c) \int_{-\rho}^{\rho} |\hat{u}^{(l)}(j, x_1)|^2 \, dx_1}_{\to 0 \text{ by } (38)} \to 0 \text{ as } l \to \infty.$$

In consequence, $\sum_{j=1}^\infty (1 + |\lambda_j^2|)\|\hat{u}^{(l)}(j, \cdot)\|^2 \to 0$ as $l \to \infty$, which contradicts our assumption and hence proves (36). Note that the lower estimate (36) finally allows to take the limit as $N \to \infty$ in the upper estimate (34). Together with (35) and the similar bounds for the two other terms of $\| \cdot \|_{H^1(\Omega_\rho)}$ this implies the claimed norm equivalence. □

For a norm equivalence corresponding to the last lemma in three dimensions we introduce the cylindrical part Σ_ρ of the boundary of Ω_ρ,

$$\Sigma_\rho := \{x \in \Omega : |\tilde{x}| = \rho\} \stackrel{m=3}{=} \{x = (\rho \cos \varphi, \rho \sin \varphi, x_3)^\top, \, \varphi \in (0, \pi), \, x_3 \in (0, H)\}. \tag{39}$$

The functions $\{\exp(in\varphi)\phi_j\}_{n\in\mathbb{Z},j\in\mathbb{N}}$ form an orthogonal basis of $L^2(\Sigma_\rho)$. Thus, we can further expand a function $u(x) = \sum_{j\in\mathbb{Z}} \hat{u}(j,\tilde{x})\phi_j(x_3) \in L^2(\Omega_\rho)$ in cylindrical coordinates as

$$u(x) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \hat{u}(j,n,r) \exp(in\varphi)\phi_j(x_3), \qquad x = \left(\begin{smallmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ x_3 \end{smallmatrix} \right), \tag{40}$$

where $\hat{u}\left(j, r\left(\begin{smallmatrix}\cos\varphi \\ \sin\varphi\end{smallmatrix}\right)\right) = \sum_{n\in\mathbb{Z}} \hat{u}(j,n,r)\exp(in\varphi)$, i.e.,

$$\hat{u}(j,n,r) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^H u(r,\varphi,x_3) \exp(in\varphi)\phi_j(x_3)\, dx_3,\, d\varphi, \qquad n \in \mathbb{Z},\, j \in \mathbb{N},\, 0 < r < \rho.$$

The transformation formula and Parseval's identity yield

$$\|u\|^2_{L^2(\Omega_\rho)} = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \int_0^\rho |\hat{u}(j,n,r)|^2\, r\, dr \qquad \text{for } u \in L^2(\Omega_\rho).$$

Lemma 4.3. *For $m = 3$ and $u \in H^1_W(\Omega_\rho)$ it holds that*

$$\|u\|^2_{H^1(\Omega_\rho)} \simeq \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \int_0^\rho \left[(1 + |\lambda_j|^2)|\hat{u}(j,n,r)|^2 + |\hat{u}'(j,n,r)|^2 + \frac{n^2}{r^2}|\hat{u}(j,n,r)|^2 \right] r\, dr. \tag{41}$$

Proof. The same density argument as in the proof of Lemma 4.2 for the two-dimensional case shows that it is sufficient to prove the above norm equivalence for arbitrary $u \in H^1_W(\Omega_\rho) \cap C^2(\overline{\Omega_\rho})$. We truncate the Fourier series of dimension three in (40) to define

$$u_N(x) = \sum_{j=1}^{N} \sum_{n=-N}^{N} \hat{u}(j,n,r) \exp(in\varphi)\phi_j(x_3), \qquad x \in \Omega_\rho,\ N \in \mathbb{N}. \tag{42}$$

As in the proof of Lemma 4.2, the orthogonality of the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ and the trigonometric monomials implies that

$$\|u\|^2_{L^2(\Omega_\rho)} = \lim_{N\to\infty} \sum_{j=1}^{N} \sum_{n=-N}^{N} \int_0^\rho |\hat{u}(j,n,r)|^2 r\, dr. \tag{43}$$

Recall the representation of the gradient in cylinder coordinates,

$$\nabla u_N = \frac{\partial u_N}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u_N}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial u_N}{\partial x_3}\boldsymbol{e}_{x_3}, \quad \text{with } \boldsymbol{e}_r = \left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{smallmatrix} \right),\ \boldsymbol{e}_\varphi = \left(\begin{smallmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{smallmatrix} \right),$$

and $\boldsymbol{e}_{x_3} = (0,0,1)^\top$. Lemma 4.1 shows that $\nabla_{\tilde{x}} u \in H^1(\{|\tilde{x}| < \rho)$, that $\|\nabla_{\tilde{x}} u\|^2_{L^2(\Omega_\rho)} = \sum_{j\in\mathbb{N}} \|\nabla_{\tilde{x}} \hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})}$, and the transformation theorem together with the orthogonality of the trigonometric polynomials implies that (see, e.g., (A.35) in [Kir11])

$$\|\nabla_{\tilde{x}} \hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} = 2\pi \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \int_0^\rho \left[\left(1 + \frac{n^2}{r^2}\right) |\hat{u}(j,n,r)|^2 + |\hat{u}'(j,n,r)|^2 \right] r\, dr. \tag{44}$$

Precisely the same arguments as in the proof of Lemma 4.2 finally also allow to prove the existence of constants $0 < c < C$ independent of u such that

$$c \sum_{j=1}^{\infty} (1+|\lambda_j|^2) \|\hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} \le \left\| \frac{\partial u_N}{\partial x_3} \right\|^2_{L^2(\Omega_\rho)} \le C \sum_{j=1}^{\infty} (1+|\lambda_j|^2) \|\hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})},$$

such that (43) yields the norm claimed equivalence. □

It is well-known that the trace operator T, first defined for continuous functions $u \in C(\overline{\Sigma_\rho})$ by $u \mapsto u|_{\Sigma_\rho}$, can be extended to a bounded linear operator from $H^1(\Omega_\rho)$ into $H^{1/2}(\Sigma_\rho)$, see, e.g., [McL00]. We will now introduce special subspaces of this trace space adapted to $H^1_W(\Omega_\rho)$. To this end, note that in the two-dimensional case the boundary $\Sigma_\rho = \Sigma_\rho^+ \cup \Sigma_\rho^-$ consists of two parts $\Sigma_\rho^\pm = \{(\pm\rho, x_2)^\top,\, x_2 \in (0,H)\}$. Thus, in the case $m = 2$ we set

$$V_2 = \left\{ v \in L^2(\Sigma_\rho) :\ v|_{\Sigma_\rho^\pm} = \sum_{j=1}^{\infty} \hat{v}^\pm(j)\phi_j,\, \sum_{j=1}^{\infty} (1+|\lambda_j|^2)^{1/2} |\hat{v}^\pm(j)|^2 < \infty \right\} \subset L^2(\Sigma_\rho)$$

with inner product defined by $(u,v)_{V_2} = \sum_{\circledast\in\pm} \sum_{j=1}^{\infty} (1+|\lambda_j|^2)^{1/2} \hat{u}^{\circledast}(j)\overline{\hat{v}(j)}$ for $u,v \in V_2$. The dual of V_2 with respect to $L^2(0,H)$ is V_2', a Hilbert space for the inner product $(u,v)_{V_2'} = \sum_{\pm} \sum_{j=1}^{\infty} (1+|\lambda_j|^2)^{-1/2} \hat{u}^\pm(j)\overline{\hat{v}(j)}$, defined for $u,v \in V_2'$. For $m = 3$, we set

$$V_3 = \Bigg\{ v(x) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \hat{v}(j,n) \exp(in\varphi)\phi_j(x_3),\ (\varphi, x_3) \in (0,2\pi)\times(0,H) : \\ \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} (1+|n|^2+|\lambda_j|^2)^{1/2} |\hat{v}(j,n)|^2 < \infty \Bigg\} \subset L^2(\Sigma_\rho)$$

with inner product $(u,v)_{V_3} := 2\pi\rho \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} (1+|n|^2+|\lambda_j|^2)^{1/2} \hat{u}(j,n)\overline{\hat{v}(j,n)}$ for $u,v \in V_3$. The dual space V_3' with respect to $L^2(\Sigma_\rho)$ of V_3 is equipped with scalar product $(u,v)_{V_3'} = 2\pi\rho \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} (1+|n|^2+|\lambda_j|^2)^{-1/2} \hat{u}(j,n,r)\overline{\hat{v}(j,n,r)}$ for $u,v \in V_3'$.

Theorem 4.4. *(Trace Operator) In dimension $m = 2$ and $m = 3$ it holds that $\|Tu\|_{V_m} \leq C\|u\|_{H^1(\Omega_\rho)}$ for all $u \in H^1_W(\Omega_\rho)$.*

Proof. Again, we prove the result merely for smooth functions u in the dense set $H^1_W(\Omega_\rho) \cap C^1(\overline{\Omega})$ and start with the two-dimensional case. The Cauchy-Schwarz inequality in $L^2(-\rho, \rho)$ gives the estimate

$$\begin{aligned}(2\rho)^2|u(j,\rho)|^2 &= \int_{-\rho}^{\rho} \frac{d}{dx_1}\big((x_1+\rho)^2|\hat{u}(j,x_1)|^2\big)\, dx_1 \\ &= 2\int_{-\rho}^{\rho}(x_1+\rho)|\hat{u}(j,x_1)|^2\, dx_1 + 2\int_{-\rho}^{\rho}(x_1+\rho)^2 \operatorname{Re}[u(n,\rho)\overline{u'(n,\rho)}]\, dx_1 \\ &\leq 4\rho\int_{-\rho}^{\rho}|\hat{u}(j,x_1)|^2\, dx_1 + 2\Big(\int_{-\rho}^{\rho}(x_1+\rho)^2|\hat{u}(j,x_1)|^2\, dx_1 \int_{-\rho}^{\rho}(x_1+\rho)^2|\hat{u}'(j,x_1)|^2\, dx_1\Big)^{1/2}\end{aligned}$$

holds. As $2ab \leq a^2 + b^2$, $(x_1+\rho)^2 \leq 4\rho^2$ for $|x_1| < \rho$, and $(1+|\lambda_j|^2)^{1/2} \leq 1 + |\lambda_j|^2$, we conclude that

$$(1+|\lambda_j|^2)^{1/2}|u(j,\rho)|^2 \leq C(\rho)(1+|\lambda_j|^2)\int_{-\rho}^{\rho}|\hat{u}(j,x_1)|^2\, dx_1 + \int_{-\rho}^{\rho}|\hat{u}'(j,x_1)|^2\, dx_1.$$

Repeating the same computation for $-\rho$ instead of ρ and summing over $j \in \mathbb{N}$ shows that

$$\begin{aligned}\|Tu\|^2_{V_2} = \|\, u|_{\Sigma_\rho}\|^2_{V_2} &= \sum_{j\in\mathbb{N}}(1+|\lambda_j|^2)^{1/2}\left[|u(j,\rho)|^2 + |u(j,-\rho)|^2\right] \\ &\leq C(\rho)\sum_{j\in\mathbb{N}}\int_{-\rho}^{\rho}\left[(1+|\lambda_j|^2)|\hat{u}(j,x_1)|^2 + |\hat{u}'(j,x_1)|^2\right]\, dx_1 \overset{(30)}{\leq} C\|u\|^2_{H^1(\Omega_\rho)}.\end{aligned}$$

In the three-dimensional case, the Cauchy-Schwarz inequality in $L^2(0,\rho)$ implies that

$$\begin{aligned}\rho^2|u(j,n,\rho)|^2 &= \int_0^{\rho}\frac{d}{dr}(r^2|u(j,n,\rho)|^2)\, dr \\ &= 2\int_0^{\rho} r|u(j,n,\rho)|^2\, dr + 2 \operatorname{Re}\int_0^{\rho} u(j,n,\rho)\overline{u'(j,n,\rho)}r^2 dr \\ &\leq 2\int_0^{\rho}|u(j,n,\rho)|^2 r\, dr + 2\rho\Big(\int_0^{\rho}|u(j,n,\rho)|^2 r\, dr\Big)^{1/2}\Big(\int_0^{\rho}|u'(j,n,\rho)|^2 r\, dr\Big)^{1/2}.\end{aligned}$$

and, as in the proof for the two-dimensional case, we find that

$$\begin{aligned}(1+|n|^2+|\lambda_j|^2)^{1/2}|u(j,n,\rho)|^2 &\leq C(1+|n|^2+|\lambda_j|^2)\int_0^{\rho}\left[|u(j,n,r)|^2 + |u'(j,n,r)|^2\right] r\, dr \\ \leq C\max(1,\rho^2)\int_0^{\rho}&\left[\left(1+\frac{|n|^2}{r^2}\right)|u(j,n,r)|^2 + (1+|\lambda_j|^2)|u(j,n,r)|^2 + |u'(j,n,r)|^2\right] r\, dr.\end{aligned}$$

Summation over $n \in \mathbb{Z}$ and $j \in \mathbb{N}$, together with (41), thus implies $\|Tu\|_{V_3} \leq C\|u\|_{H^1(\Omega_\rho)}$. $\square$

Since $V'_{2,3}$ are the dual spaces to $V_{2,3}$ for the pivot space $L^2(\Sigma_\rho)$ we abbreviate

$$\langle u, v\rangle = (u,v)_{V'_m \times V_m} \qquad \text{for } u \in V'_m \text{ and } v \in V_m,\ m = 2, 3.$$

Note that the definition of V_m and V'_m and the orthogonality of the basis functions $\{\phi_j\}_{j\in\mathbb{N}}$ and $\{\exp(in\,\cdot)\}_{n\in\mathbb{Z}}$ implies that

$$|\langle u, v\rangle| = \left|(u,v)_{V'_m\times V_m}\right| \le \begin{cases} \sum_{j=1}^{\infty} \hat{u}(j,x_1)\overline{\hat{v}}(j,x_1) & \text{for } m = 2,\\ 2\pi\rho \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}} \overline{\hat{u}(j,n,r)}\hat{v}(j,n,r) & \text{for } m = 3. \end{cases}$$

This shows that $|(u,v)_{V'_m\times V_m}| \le \|u\|_{V'_m}\|v\|_{V_m}$ for all $u \in V'_m$ and $v \in V_m$.

5 The 3D Exterior Dirichlet-to-Neumann Operator

Our aim is now to determine a exterior Dirichlet-to-Neumann map on the surface Σ_ρ that maps Dirichlet boundary values in V_m to the Neumann boundary values of the (unique) radiating solution in $\Omega \setminus \Omega_\rho$ to the Helmholtz equation (1). We establish properties of this mapping first in three dimensions, before treating to the (easier) two-dimensional case in the next section.

As above, we assume that Assumption 3.1 holds, i.e., no eigenvalue $\lambda_j^2 \in \mathbb{R}$ vanishes. Thus, in the three-dimensional case, i.e., $m = 3$, all terms in following series are well-defined,

$$u(x) = \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}} \hat{u}(j,n) H_n^{(1)}(i\lambda_j r)\exp(in\varphi)\phi_j(x_3), \qquad x = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3\end{pmatrix} \in \Omega\setminus\Omega_\rho. \tag{45}$$

Hence, u defines a formal solution to the Helmholtz equation (1) in $\Omega\setminus\Omega_\rho$ that satisfies the waveguide boundary conditions $u(x) = 0$ for $x_3 = 0$ and $\partial u/\partial\nu = 0$ for $x_3 = H$. Indeed, the Hankel function $H_n^{(1)}$ of the first kind and order n satisfies Bessel's differential equation such that

$$\tilde{x} \mapsto v_{j,n}(\tilde{x}) = H_n^{(1)}(i\lambda_j r)\exp(in\varphi), \qquad \tilde{x} = r\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix},\ r > 0,$$

satisfies the two-dimensional Helmholtz equation $(\Delta_{\tilde{x}} - \lambda_j^2)v_{n,j} = 0$ in $\mathbb{R}^2\setminus\{0\}$, see [CK12]. Thus, the results from Section 2 on solutions to (1) via separation of variables imply that each term in the series (45) solves the Helmholtz equation (1). An asymptotic expansion of $H_n^{(1)}$ for large arguments moreover shows that each term of u satisfies either the radiation condition (18) or the boundedness condition (19), i.e., u is a radiating solution to the Helmholtz equation. We can formally compute the normal derivative on Σ_ρ as

$$\frac{\partial u}{\partial r}(x) = i\sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}} \lambda_j \hat{u}(j,n) H_n^{(1)\prime}(i\lambda_j\rho)\exp(in\varphi)\phi_j(x_3) \qquad \text{for } x = \begin{pmatrix}\rho\cos\varphi\\ \rho\sin\varphi\\ x_3\end{pmatrix} \in \Sigma_\rho.$$

This formula motivates the following definition.

Definition 5.1. *(Dirichlet-to-Neumann operator) For $\psi \in V_3$ with corresponding Fourier coefficients $\hat{\psi}(j,n)$ we formally define the Dirichlet-to-Neumann operator Λ by*

$$\Lambda\psi(x) := i\sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}\lambda_j \frac{H_n^{(1)'}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)}\hat{\psi}(j,n)\exp(in\varphi)\phi_j(x_3) \qquad \textit{for } x = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ x_3 \end{pmatrix} \in \Sigma_\rho. \tag{46}$$

The following result shows that the Dirichlet-to-Neumann operator Λ is well-defined and bounded from V_3 into V_3'.

Lemma 5.2. *(1) The Dirichlet-to-Neumann operator Λ defined in* (46) *is well-defined and bounded from V_3 into V_3', i.e., $\|\Lambda\psi\|_{V_3'} \le C\|\psi\|_{V_3}$ for $\psi \in V_3$ and some constant $C > 0$.*

(2) If $\psi \in V_3$, then the function

$$u(x) = \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}\hat{\psi}(j,n)\frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j\rho)}\exp(in\varphi)\phi_j(x_3), \qquad x = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3 \end{pmatrix} \in \Omega\setminus\overline{\Omega_\rho}, \tag{47}$$

belongs to $H^1_{\mathrm{loc}}(\Omega\setminus\overline{\Omega_\rho})$ and there is $C = C(\rho) > 0$ independent of ψ such that $\|u\|_{H^1_{\mathrm{loc}}(\Omega\setminus\overline{\Omega_\rho})} \le C\|\psi\|_{V_3}$. Further, u is a weak solution to the Helmholtz equation

$$\Delta u + \frac{\omega^2}{c^2(x_m)}u = 0 \quad \textit{in } \Omega\setminus\overline{\Omega_\rho} \tag{48}$$

and satisfies both the waveguide boundary conditions $u(x) = 0$ for $x_3 = 0$ and $\partial u/\partial\nu = 0$ for $x_3 = H$ and the radiation and boundedness conditions (24).

Proof. (1) For simplicity, we introduce the auxiliary coefficients

$$\hat{w}(j,n) = \begin{cases} \hat{\psi}(j,n) & \text{for } n = 0, \\ (|\hat{\psi}(j,n)|^2 + |\hat{\psi}(j,-n)|^2)^{-1/2} & \text{for } n \neq 0. \end{cases} \tag{49}$$

Due to the relation $H_{-n}(z) = (-1)^n H_n(z)$ for $z \neq 0$ from (9.1.5) in [AS64], we compute that

$$\frac{H_n^{(1)'}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)} = \frac{H_{-n}^{(1)'}(i\lambda_j\rho)}{H_{-n}^{(1)}(i\lambda_j\rho)}.$$

Assume now that $\psi \in V_3$ is given by $\psi = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \hat{\psi}(j,n) \exp(in\varphi)\phi_j(x_3)$. By the definition of the Dirichlet-to-Neumann operator and the Fourier coefficients $\hat{w}(j,n)$ we compute

$$\begin{aligned}
\|\Lambda\psi\|_{V_3'}^2 &= \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} (1+|n|^2+|\lambda_j|^2)^{-1/2} \left| \lambda_j \hat{\psi}(j,n) \frac{H_n^{(1)'}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)} \right|^2 \\
&= \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} (1+|n|^2+|\lambda_j|^2)^{-1/2} \hat{w}(j,n)^2 \left| \lambda_j \frac{H_n^{(1)'}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)} \right|^2 .
\end{aligned}$$

Note again that all terms are well-defined since $\lambda_j \neq 0$ for all $j \in \mathbb{N}$ by Assumption 3.1 and since $|H_n^{(1)}(z)|$ is strictly positive for $z > 0$ and for $z \in \mathbb{C}$ with positive imaginary part (recall that either λ_j or $i\lambda_j$ are strictly positive). From the Appendix of [AGL08] we know that

$$\frac{H_n^{(1)'}(z)}{H_n^{(1)}(z)} = \frac{H_{|n|-1}^{(1)}(z)}{H_{|n|}^{(1)}(z)} - \frac{|n|}{z} \qquad \text{for } z \in \mathbb{C} \text{ with } \arg(z) \in (-\pi/2, \pi/2]. \tag{50}$$

This equality allows to estimate

$$\|\Lambda\psi\|_{V_3'}^2 \leq C(\rho) \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} (1+|n|^2+|\lambda_j|^2)^{-1/2} |\lambda_j|^2 |\hat{w}(j,n)|^2 \left(\left| \frac{H_{n-1}^{(1)}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)} \right|^2 + \frac{n^2}{|\lambda_j|^2\rho^2} \right).$$

We estimate each part of the sum on the right-hand side of the last equation separately. Due to the definition of the Fourier coefficients $\hat{w}(j,n)$ in (49) we have

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{N}_0} (1+|n|^2+|\lambda_j|^2)^{-1/2} |\lambda_j|^2 |\hat{w}(j,n)|^2 \frac{n^2}{|\lambda_j|^2\rho^2} \\
&\qquad\qquad \leq \frac{1}{\rho^2} \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} (1+|n|^2+|\lambda_j|^2)^{-1/2} n^2 |\hat{\psi}(j,n)|^2 \leq \frac{1}{\rho^2} \|\psi\|_{V_3}^2 .
\end{aligned}$$

Next, by Lemma A.2 in [AGL08] we know that for $z \in \mathbb{C}$ such that $|z| \geq \rho > 0$ and $\arg(z) \in (-\pi/2, \pi/2]$ and $n \in \mathbb{N}$ there holds

$$\left| \frac{H_{n-1}^{(1)}(z)}{H_n^{(1)}(z)} \right| \leq C(\rho), \qquad \text{for } |z| \geq \rho > 0,\ \arg(z) \in (-\pi/2, \pi/2],\ n \in \mathbb{Z}. \tag{51}$$

We recall from (10) that $\lambda_j^2 \le Cj^2$. In particular, we find that

$$\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(1+|n|^2+|\lambda_j|^2)^{-1/2}|\lambda_j\hat{w}(j,n)|^2\Big|\frac{H^{(1)}_{n-1}(i\lambda_j\rho)}{H^{(1)}_n(i\lambda_j\rho)}\Big|^2$$
$$\le C(\rho)\sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}(1+|n|^2+|\lambda_j|^2)^{-1/2}|\lambda_j|^2|\hat{w}(j,n)|^2 \le C(\rho)\|\psi\|^2_{V_3}.$$

(2) In this part we abbreviate the domain $\Omega_R\backslash\overline{\Omega_\rho}$ for $R>\rho$ as $\Omega_{\rho,R}$ and the corresponding two-dimensional domain $\{\rho<|\tilde{x}|<R\}\subset\mathbb{R}^2$ by $\tilde{\Omega}_{\rho,R}$. For $|\tilde{x}|>\rho$, the function u from (47) can be written as

$$u(x)=\sum_{j\in\mathbb{N}}\hat{u}(j,\tilde{x})\phi_j(x_3)\quad\text{with } \hat{u}(j,\tilde{x})=\sum_{n\in\mathbb{Z}}\hat{\psi}(j,n)\frac{H^{(1)}_n(i\lambda_j r)}{H^{(1)}_n(i\lambda_j\rho)}\exp(in\varphi),\quad \tilde{x}=\left(\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi\end{smallmatrix}\right).$$

We will first show that the latter series converges in $H^1(\Omega_{\rho,R})$ for arbitrary $R>\rho$, such that $u\in H^1_{\mathrm{loc}}(\Omega\setminus\overline{\Omega_\rho})$. Since $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,H)$ we note, as in the proof of Lemma 32

$$\|u\|^2_{L^2(\Omega_{\rho,R})}+\|\nabla_{\tilde{x}}u\|^2_{L^2(\Omega_{\rho,R})}\le\sum_{j\in\mathbb{N}}\|\hat{u}(j,\cdot)\|^2_{H^1(\tilde{\Omega}_{\rho,R})}\|\phi_j\|^2_{L^2(0,H)}\le\sum_{j\in\mathbb{N}}\|\hat{u}(j,\cdot)\|^2_{H^1(\tilde{\Omega}_{\rho,R})}.$$

Moreover, the proof of Theorem 4.3 shows that

$$\|\partial u/\partial x_3\|^2_{L^2(\Omega_{\rho,R})}\le C\sum_{j\in\mathbb{N}}(1+|\lambda_j|^2)\|\hat{u}(j,\cdot)\|^2_{L^2(\tilde{\Omega}_{\rho,R})}.$$

Thus,

$$\|u\|^2_{H^1(\Omega_{\rho,R})}\le C\sum_{j\in\mathbb{N}}\left[\|\hat{u}(j,\cdot)\|^2_{H^1(\tilde{\Omega}_{\rho,R})}+(1+|\lambda_j|^2)\|\hat{u}(j,\cdot)\|^2_{L^2(\tilde{\Omega}_{\rho,R})}\right].\tag{52}$$

For $\xi\in\mathbb{R}$ and a parameter $k^2=1+\max_{j\in\mathbb{N}}(-\lambda_j^2)<\infty$ we set

$$\alpha(\xi)=\begin{cases}\sqrt{k^2-\xi^2} & \text{if } k^2\ge\xi^2,\\ i\sqrt{\xi^2-k^2} & \text{if } k^2\ge\xi^2.\end{cases}$$

The latter function is then used to define

$$\tilde{v}_{n,\xi}(\tilde{x}):=\frac{H^{(1)}_n(r\alpha(\xi))}{H^{(1)}_n(\rho\alpha(\xi))}\exp(in\varphi)\qquad\text{for } \tilde{x}=r\left(\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right)\in\tilde{\Omega}_{\rho,R} \text{ and } n\in\mathbb{Z}.$$

It is not difficult to see that $\tilde{v}_{n,\xi}$ belongs to $H^1(\tilde{\Omega}_{\rho,R})$. Moreover, Lemma A6 in [CH07] states that there exists $C > 0$ independent of $\xi \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that

$$\|\tilde{v}_{n,\xi}\|^2_{H^1(\tilde{\Omega}_{\rho,R})} \le C(\rho, R)\,(1 + n^2 + \xi^2)^{1/2}.$$

Since $k^2 = 1 + \max_{j\in\mathbb{N}}(-\lambda_j^2)$ it holds $k^2 + \lambda_j^2 > 1$ for all $j \in \mathbb{N}$ such that there exists a unique positive solution $\xi_j > 0$ to $\xi_j^2 = k^2 + \lambda_j^2$. Note that the latter equation implies that $\alpha(\xi_j) = i\lambda_j$ and that $\xi_j^2 \le C(k)(1 + |\lambda_j|^2)$. Thus,

$$\begin{aligned}\|\tilde{v}_{n,\xi_j}\|^2_{H^1(\tilde{\Omega_{\rho,R}})} &\le C(\rho, R)\,(1 + n^2 + \xi_j^2)^{1/2}\\ &= C(\rho, R)\,(1 + n^2 + k^2 + \lambda_j^2)^{1/2} \le C(\rho, R, k)\,(1 + n^2 + |\lambda_j|^2)^{1/2}.\end{aligned}$$

Since the trigonometric monomials $\varphi \mapsto \exp(in\varphi)$ are orthogonal on $(0, 2\pi)$, and since their derivatives are $\varphi \mapsto in\exp(in\varphi)$, the functions $\tilde{v}_{n,\xi_j}$ are orthogonal for the inner product of $H^1(\tilde{\Omega}_{\rho,R})$. Thus, the last estimate implies that

$$\begin{aligned}\|\hat{u}(j,\cdot)\|^2_{H^1(\tilde{\Omega}_{\rho,R})} &= \Big\|\sum_{n\in\mathbb{Z}} \hat{\psi}(j,n)\tilde{v}_{n,\xi_j}\Big\|^2_{H^1(\tilde{\Omega}_{\rho,R})}\\ &\le \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 \|\tilde{v}_{n,\xi_j}\|^2_{H^1(\tilde{\Omega}_{\rho,R})}\\ &\le C(\rho, R, k)\sum_{n\in\mathbb{Z}}(1 + n^2 + |\lambda_j|^2)^{1/2}|\hat{\psi}(j,n)|^2.\end{aligned}$$

Of course, the corresponding L^2-estimate holds as well due to orthogonality,

$$\|\hat{u}(j,\cdot)\|^2_{L^2(\tilde{\Omega}_{\rho,R})} = \Big\|\sum_{n\in\mathbb{Z}} \hat{\psi}(j,n)\tilde{v}_{n,\xi_j}\Big\|^2_{L^2(\tilde{\Omega}_{\rho,R})} \le \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 \|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})}.$$

In consequence, (52) shows that

$$\|u\|^2_{H^1(\Omega_{\rho,R})} \le C\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}\left[(1 + n^2 + |\lambda_j|^2)^{1/2} + (1 + |\lambda_j|^2)\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})}\right]|\hat{\psi}(j,n)|^2.$$

The L^2-norm of $\tilde{v}_{n,\xi_j}$ is finally easy to estimate: First, for all $j \in \mathbb{N}$,

$$\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})} = 2\pi\int_\rho^R \left|\frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j\rho)}\right|^2 \frac{dr}{r} \le \frac{2\pi}{\rho}\int_\rho^R \frac{\left|H_n^{(1)}(i\lambda_j r)\right|^2}{\left|H_n^{(1)}(i\lambda_j\rho)\right|^2}\,dr \le 2\pi(R-\rho)$$

since $\left|H_n^{(1)}(i\lambda_j r)\right|^2/\left|H_n^{(1)}(i\lambda_j\rho)\right| \le 1$ for $r > 0$ by Lemma A2 in [CH07]. Moreover, if $j > J$, i.e., if $\lambda_j^2 > 0$, then $\left|H_n^{(1)}(i\lambda_j r)\right|^2/\left|H_n^{(1)}(i\lambda_j\rho)\right|^2 \le \exp(-(r-\rho)|\lambda_j|)$ for $r > \rho$ due to Lemma A3 in [CH07] and

$$\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{\Omega}_{\rho,R})} \le \frac{2\pi}{\rho}\int_\rho^R e^{-(r-\rho)|\lambda_j|}\,dr \le \frac{2\pi}{|\lambda_j|}(1-\exp(-(R-\rho)|\lambda_j|)) \le \frac{4\pi}{|\lambda_j|} \le \frac{C}{(1+|\lambda_j|^2)^{1/2}}.$$

This shows that

$$\|u\|^2_{H^1(\Omega_{\rho,R})} \le C\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}(1+n^2+|\lambda_j|^2)^{1/2}|\hat{\psi}(j,n)|^2 = C\|\psi\|^2_{V_3}.$$

The function u satisfies the Helmholtz equation (48) weakly by construction of the eigenfunctions ψ_j to (8), since $\tilde{v}_{n,\xi_j}$ solves $(\Delta_{\tilde{x}} - \lambda_j^2)\tilde{v}_{n,\xi_j} = 0$ in $\{|\tilde{x}| > \rho\}$ and since the series in (47) was shown to converge in $H^1(\Omega_{\rho,R})$. The same argument shows that u satisfies the waveguide boundary conditions. Well-known properties of Hankel and Kelvin functions show that $\tilde{v}_{n,\xi_j}$ is a radiating solution to the Helmholtz equation if $1 \le j \le J$, i.e., $\lambda_j^2 < 0$, whereas $\tilde{v}_{n,\xi_j}$ is bounded (and even exponentially decaying) if $j \ge J$, i.e., $\lambda_j^2 > 0$. This implies that u satisfies the radiation and boundedness conditions (24). □

The next lemma formulates a weak coercivity result for Λ when applied to $\psi \in V_3$ with representation $\psi = \sum_{j,n}\hat{\psi}(j,n)\exp(in\,\cdot)\phi_j$.

Lemma 5.3. *There exist constants $C > 0$ and $c > 0$ such that Λ is L^2-coercive at small frequencies: For $0 < \omega \le C$ it holds that*

$$-\langle\Lambda\psi,\psi\rangle \ge c\omega\|\psi\|^2_{L^2(\Sigma_\rho)} \qquad \textit{for all } \psi \in V_3.$$

Proof. Due to (10) we can choose $\omega > 0$ so small such that the lower bound $(\pi(2j-1)/(2H))^2 - \omega^2/c_-^2$ of all eigenvalues λ_j^2 is positive. With this choice, all roots λ_j are positive, too, i.e., all waveguide modes are evanescent. For $\psi \in V_3$ we use again the auxiliary coefficients $\hat{w}(j,n)$ defined in (49) and note that

$$\frac{\langle\Lambda\psi,\psi\rangle}{2\pi\rho} = i\sum_{j=1}^\infty\sum_{n=0}^\infty\lambda_j\frac{H_n^{(1)'}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)}|\hat{w}(j,n)|^2 \overset{(50)}{=} i\sum_{j=1}^\infty\sum_{n=0}^\infty\lambda_j\Big(\frac{H_{n-1}^{(1)}(i\lambda_j\rho)}{H_n^{(1)}(i\lambda_j\rho)} - \frac{n}{i\lambda_j\rho}\Big)|\hat{w}(j,n)|^2.$$

Note that the argument of all Hankel functions in the last expression is purely imaginary since $\lambda_j > 0$ for all $j \in \mathbb{N}$. Thus, we can reformulate the last expression using the modified Bessel functions K_n,

$$\frac{H_{n-1}^{(1)}(z)}{H_n^{(1)}(z)} = i\frac{K_{n-1}^{(1)}(|z|)}{K_n^{(1)}(|z|)} \qquad \text{for } z \in i\mathbb{R}_{>0},\ n \in \mathbb{Z}.$$

Note that the modified Bessel functions K_n satisfy

$$\frac{K_{-1}(t)}{K_0(t)} \geq 1, \qquad \frac{K_0(t)}{K_1(t)} \geq 1 - \frac{2}{t}, \qquad \text{and} \qquad 1 \geq \frac{K_n(t)}{K_{n+1}(t)} \geq \frac{t}{t+2n}, \qquad t > 0, \tag{53}$$

which is shown in [AGL08, Lemma A.1]. Moreover, $K_0(t) > 0$ and $K_1(t) > 0$ for $t > 0$ such that the lower bound $K_0(t)/K_1(t) \geq 1 - 2/t$ ensures the existence of a small constant $c_1 > 0$ such that $K_0(t)/K_1(t) \geq c_1$ holds for all $t \geq \lambda_1 \rho$. The increasing order $0 < \lambda_1^2 \leq \lambda_2^2 \leq \dots$ hence implies that $K_0(\lambda_j \rho)/K_1(\lambda_j \rho) \geq c_1$ for all $j \in \mathbb{N}$. Using this bound together with those from (53), we obtain that

$$\begin{aligned} -\langle \Lambda\psi, \psi \rangle &= 2\pi\rho \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \Big(\lambda_j \frac{K_{n-1}^{(1)}(\lambda_j \rho)}{K_n^{(1)}(\lambda_j \rho)} + \frac{n}{\rho} \Big) |\hat{w}(j,n)|^2 \\ &\geq 2\pi\rho \sum_{j=1}^{\infty} \Big[\lambda_j |w(j,0)|^2 + c_1 \lambda_j |\hat{w}(j,1)|^2 + \sum_{n=2}^{\infty} \lambda_j \big(\frac{\lambda_j \rho}{\lambda_j \rho + 2n} + \frac{n}{\lambda_j \rho} \big) | \Big] \hat{w}(j,n)|^2. \end{aligned}$$

Note that the binomial formula yields that

$$\lambda_j \left(\frac{\lambda_j \rho}{\lambda_j \rho + 2n} + \frac{n}{\lambda_j \rho} \right) \geq 2\lambda_j \left(\frac{n}{\lambda_j \rho + 2n} \right)^{1/2} \geq 2\lambda_j \left(\frac{1}{\lambda_j \rho + 2} \right)^{1/2} \geq 2 \left(\frac{\lambda_1}{\lambda_1 \rho + 2} \right)^{1/2} \lambda_j^{1/2} > 0,$$

because $n \mapsto n/(\lambda_j \rho + 2n)$ and $j \mapsto \lambda_1/(\lambda_1 \rho + 2)$ increase in n and j, respectively. By Lemma 2.2(a) it holds for $j \in \mathbb{N}$ and $0 < \omega < \min\{(\pi c_-)/(4H), 1\}$ that

$$\frac{3}{c_-^2} \omega^2 \leq \omega^2 \left(\frac{\pi^2}{4\omega^2 H^2} (2j-1)^2 - \frac{1}{c_-^2} \right) \leq \lambda_j^2. \tag{54}$$

Thus, for $\omega > 0$ small enough and $c_* = 3/c_-^2$ it holds that $\lambda_j^2 \geq c_* \omega^2$ for all $j \in \mathbb{N}$. Monotonicity of the square root function hence directly implies that $\lambda_j \geq c_*^{1/2} \omega$ and $\lambda_j^{1/2} \geq c_*^{1/4} \omega^{1/2} \geq c_*^{1/4} \omega$ since ω is smaller than one. If we further choose $\omega > 0$ such that $\omega < \min(c_*^{-1/2}, (\pi c_-)/(2H))$ then $\lambda_j^2 < 1$ due to (54), i.e., $\lambda_j \leq \sqrt{\lambda_j}$. In consequence, we obtain for some $c > 0$ that

$$-\langle \Lambda\psi, \psi \rangle \geq c \sum_{j=1}^{\infty} \left[\lambda_j |w(j,0)|^2 + \lambda_j |\hat{w}(j,1)|^2 + \sqrt{\lambda_j} \sum_{n=2}^{\infty} |\hat{w}(j,n)|^2 \right] \geq c\omega \sum_{j=1}^{\infty} \sum_{n \in \mathbb{Z}} |\hat{\psi}(j,n)|^2.$$

□

Lemma 5.4. *If $\omega > 0$ then there exists $C = C(\omega) > 0$ such that for all $\psi \in V_3$ with Fourier coefficients $\hat{\psi}(j,n)$ it holds that*

$$-\operatorname{Re}\langle \Lambda\psi, \psi\rangle \geq -2\pi\rho\, C \sum_{j=1}^{J} \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 = -C\|\psi\|^2_{L^2(\Sigma_\rho)},$$

where $J = J(\omega, c, H)$ denotes the number of propagating waveguide modes.

Proof. We use again the auxiliary coefficients $\hat{w}(j,n)$ defined in (49) and, by the same arguments as in the proof of Lemma 5.3, compute that

$$-\operatorname{Re}\langle \Lambda\psi, \psi\rangle = 2\pi\rho \operatorname{Re} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} i\lambda_j \Big(\frac{n}{i\lambda_j\rho} - \frac{H^{(1)'}_{n-1}(i\lambda_j\rho)}{H^{(1)}_n(i\lambda_j\rho)} \Big) |\hat{w}(j,n)|^2.$$

Further, since for $j > J$ all eigenvalues λ_j are positive, the arguments of the proof of Lemma 5.3 show that for $j > J$ all terms in the series are positive. Omitting these terms we obtain that

$$\begin{aligned} -\operatorname{Re}\langle \Lambda\psi, \psi\rangle &\geq 2\pi\rho\, \operatorname{Re} \sum_{j=1}^{J} \sum_{n=0}^{\infty} i\lambda_j \left(\frac{n}{i\lambda_j\rho} - \frac{H^{(1)'}_{n-1}(i\lambda_j\rho)}{H^{(1)}_n(i\lambda_j\rho)} \right) |\hat{w}(j,n)|^2 \\ &\geq -2\pi\rho \sum_{j=1}^{J} \sum_{n=0}^{\infty} i\lambda_j \left| \frac{H^{(1)'}_{n-1}(i\lambda_j\rho)}{H^{(1)}_n(i\lambda_j\rho)} \right| |\hat{w}(j,n)|^2, \end{aligned}$$

because $i\lambda_j > 0$ for $j = 1, \dots, J$. Since the finite set of numbers $\{i\lambda_j\rho\}_{j=1}^J \subset \mathbb{R}$ is bounded away from zero, estimate (51) implies that

$$\left| \frac{H^{(1)'}_{n-1}(i\lambda_j\rho)}{H^{(1)}_n(i\lambda_j\rho)} \right| \leq C \qquad \text{for } j = 1, \dots, J.$$

The constant C in the last estimate depends on ω, ρ, and of course also on the waveguide setting. Therefore, we finally deduce that

$$-\operatorname{Re}\langle \Lambda\psi, \psi\rangle \geq -C \sum_{j=1}^{J} \sum_{n=0}^{\infty} |\hat{w}(j,n)|^2 = -C \sum_{j=1}^{J} \sum_{n\in\mathbb{Z}} |\hat{\psi}(j,n)|^2 \geq -C\|\psi\|^2_{L^2(\Sigma_\rho)}.$$

$\square$

To be able to apply analytic Fredholm theory when establishing existence theory for the scattering problem (22–24) we finally show that $\Lambda = \Lambda_\omega$ depends analytically (i.e., holomorphically) on the frequency ω.

Lemma 5.5. *For all $\omega_* > 0$ such that $\lambda_j(\omega_*) \neq 0$ for $j \in \mathbb{N}$ and all $\omega^* > 0$ small enough to satisfy the assumption of Lemma 5.3 there exists an open connected set $U \subset \mathbb{C}$ containing ω_* and ω^* such that $\omega \mapsto \Lambda_\omega$ is an analytic operator-valued function in U.*

Proof. Due to Theorem 8.12(b) in [Muj85] we merely need to show that

$$\langle \Lambda u, v\rangle = 2\pi i\rho \sum_{j=1}^{\infty} \lambda_j(\omega) \sum_{n\in\mathbb{Z}}^{\infty} \frac{H_n^{(1)'}(i\lambda_j(\omega)\rho)}{H_n^{(1)}(i\lambda_{\ell_j(\omega)}(\omega)\rho)} \hat{u}(j,n)\overline{\hat{v}(j,n)} \tag{55}$$

$$= 2\pi i\rho \sum_{j=1}^{\infty} \lambda_{\ell_j(\omega)}(\omega) \sum_{n\in\mathbb{Z}}^{\infty} \Big[\frac{H_{n-1}^{(1)'}(i\lambda_{\ell_j(\omega)}(\omega)\rho)}{H_n^{(1)}(i\lambda_{\ell_j(\omega)}(\omega)\rho)} - \frac{n}{i\lambda_{\ell_j(\omega)}(\omega)\rho}\Big] \hat{u}(j,n)\overline{\hat{v}(j,n)}, \quad u,v \in V_3'$$

is a holomorphic function in an open connected set $U \subset \mathbb{C}$ that satisfies the properties claimed in the lemma. From Lemma 2.4 we know that all eigenvalue functions $\omega \mapsto \lambda_j^2(\omega)$ can be extended to some open neighborhood U_0 of $\mathbb{R}_{>0}$. We choose $\delta_1 > 0$ such that $U_1 = \{z \in U,\, 0 \le \operatorname{Re}(z) \le \omega^* + 1,\, |\operatorname{Im}(z)| \le \delta_1\} \subset U$ is connected, compact and contains ω_* and ω^*. Due to Theorem 2.4, the set $K_0 = \{\omega \in U_1, \text{ there is } j \in \mathbb{N} \text{ such that } \lambda_j(\omega)^2 = 0\}$ is finite. Thus, by further reducing the parameter δ_1 we can assume without loss of generality that K_0 contains merely real numbers. Recall that the square root function $z \mapsto z^{1/2}$ that was defined for complex numbers via a branch cut at the positive real axis is holomorphic in the slit complex plane $\mathbb{C}\backslash i\mathbb{R}_{\ge 0}$. The the roots $\omega \mapsto \lambda_{\ell_j(\omega)}(\omega)$ are hence holomorphic functions in the set $U_2 := \{z \in U_1, \operatorname{Im} z < 0 \text{ if } \omega \in K_0\}$. Further restricting this set we define the open set $U_3 := \{z \in U_2,\, B(z,\delta_2) \subset U_2\}$ for a parameter $\delta_2 > 0$. For δ_2 small enough U_3 is open, connected and contains ω_* and ω^*. Recall that the Hankel function $z \mapsto H_n^{(1)}(z)$ and its derivative are holomorphic in the domain $\{z \in \mathbb{C},\, z \neq 0,\, -\pi/2 < \arg(z) < \pi\}$. The fraction $z \mapsto H_n^{(1)'}(z)/H_n^{(1)}(z)$ is holomorphic for $z \neq 0$ and $\arg(z) \in [0,\pi)$ since $z \mapsto H_n^{(1)}(z)$ does not possess zeros in this domain. Moreover, an infinite number of zeros of $z \mapsto H_n^{(1)}(z)$ in the lower complex half-plane is contained in the quadrant $-\pi < \arg(z) \le -\pi/2$, while at most n zero are contained in $-\pi/2 < \arg(z) \le 0$, compare the paragraph on complex zeros of the Hankel function in [AS64, pg. 373–374]. If follows from [CS82, eq. (2.8)] or [AS64, pg. 374] that these finitely many zeros lie in the sector $-\pi/2 < \arg(z) \le -\epsilon$ for some $\epsilon > 0$, independent of n, i.e., $z \mapsto H_n^{(1)'}(z)/H_n^{(1)}(z)$ is holomorphic in $\{z \neq 0,\, \arg(z) \in (-\epsilon, \pi+\epsilon)\}$. Since the numbers $i\lambda_j$ are either positive or purely imaginary with positive imaginary part we deduce that, upon reducing the parameter $\delta_1 > 0$ for the construction of $U_{1,2,3}$ a second time, the function $\omega \mapsto H_n^{(1)'}(i\lambda_{\ell_j(\omega)}(\omega)\rho)/H_n^{(1)}(i\lambda_{\ell_j(\omega)}(\omega)\rho)$ is holomorphic for $\omega \in U_3$.

Thus, each term in the series in (55) is holomorphic in U_3 and can hence be developed locally into a power series in ω. Holomorphy of the entire series follows from the uniform and absolute convergence of this series: If we set

$$g_j(\omega) = \lambda_j(\omega) \sum_{n\in\mathbb{Z}} \Big[\frac{H^{(1)}_{n-1}(i\lambda_j(\omega)\rho)}{H^{(1)}_n(i\lambda_j(\omega)\rho)} - \frac{n}{i\lambda_j(\omega)\rho}\Big] \hat{u}(j,n)\overline{\hat{v}(j,n)} \tag{56}$$

$$= \lambda_j(\omega) \sum_{n\in\mathbb{Z}} \frac{H^{(1)}_{n-1}(i\lambda_j(\omega)\rho)}{H^{(1)}_n(i\lambda_j(\omega)\rho)} \hat{u}(j,n)\overline{\hat{v}(j,n)} - R_j(u,v), \quad R_j(u,v) := \sum_{n\in\mathbb{Z}} \frac{n}{i\rho} \hat{u}(j,n)\overline{\hat{v}(j,n)},$$

then $R_j(u,v)$ is a bounded sesquilinear form on V_3 independent of ω. For all $j > J(\omega^* + 1, c, H)$ it holds that $i\lambda_j(\omega) \in i\mathbb{R}_{>0}$ for all $\omega \in U_3 \cap \mathbb{R}$ such that

$$\left| \frac{H^{(1)}_{n-1}(i\lambda_j(\omega)\rho)}{H^{(1)}_n(i\lambda_j(\omega)\rho)} \right| = \left| \frac{K_{n-1}(\lambda_j(\omega)\rho)}{K_n(\lambda_j(\omega)\rho)} \right| \le C \qquad \text{for } \omega \in U_3,\ n \in \mathbb{Z},$$

due to [AGL08, Lemma A.2 & (A10)]. For $1 \le j \le J(\omega^* + 1, c, H)$ the asymptotic expansion of the Hankel functions for large orders, see [AS64, (9.3.1)], implies that there is a constant $C > 0$ such that the last bound is uniformly valid for all $j \in \mathbb{N}$. Thus,

$$|g_j(\omega)| \le \sum_{n\in\mathbb{Z}} (C|\lambda_j(\omega)| + n/\rho)\, |\hat{u}(j,n)\hat{v}(j,n)| \le C\|u\|_V \|v\|_V \tag{57}$$

since $\omega \mapsto \lambda_j(\omega)$ is holomorphic on U_0 and hence in particular bounded on the compactly embedded subset U_3. We deduce that the series in (56) converges absolutely and uniformly for each $\omega \in U_3$. Since the analytic dependence of each series term on ω implies that each term can locally be represented by its convergent Taylor series with coefficients $d^{(j)}_l(u,v)$ that yield bounded sesquilinear forms,

$$g_j(\omega) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\infty} d^{(j)}_l(u,v)(\omega - \omega^*)^l, \qquad \omega \in U_3.$$

Uniform convergence of the series in $n \in \mathbb{Z}$ implies that the two limits in n and l can be interchanged. Thus, g_j has a convergent Taylor expansion as well and is hence a holomorphic function of $\omega \in U_3$.

As in the proof of Lemma 5.2 one shows that the series of the exterior Dirichlet-to-Neumann operator Λ in (55) is also uniformly convergent in $j \in \mathbb{N}$, such that the Taylor series expansion of g_j can again be interchanged with the series in $j \in \mathbb{N}$. This finally implies the claim of the lemma. $\square$

6 The 2D Exterior Dirichlet-to-Neumann Operator

In this section we study the exterior Dirichlet-to-Neumann operator for a two-dimensional setting. The derivation of its representation and the results on its boundedness, coercivity and analyticity are rather analogous and usually easier to prove than in the three-dimensional case treated in the last section. For this reason we will announce the corresponding results in this section and merely indicate where the proofs differ from those in dimension three.

As in the last section we assume that Assumption 3.1 holds, i.e., no eigenvalue $\lambda_j^2 \in \mathbb{R}$ vanishes. In the case $m = 2$, solutions to the Helmholtz equation (1) in $\Omega \setminus \Omega_\rho$ that satisfy the waveguide boundary conditions $u(x) = 0$ for $x_3 = 0$ and $\partial u/\partial \nu = 0$ for $x_3 = H$ gained by separation of variables take the form

$$u(x) = \sum_{j=1}^{\infty} \hat{u}(j) \exp(-\lambda_j |x_1|) \phi_j(x_2), \qquad x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \Omega \setminus \Omega_\rho, \text{ i.e., } |x_1| > \rho.$$

Hence, u defines a formal solution to the Helmholtz equation (1) in $\Omega \setminus \Omega_\rho$. If $\lambda_j^2 > 0$ then $i\lambda_j \in i\mathbb{R}_{>0}$ and the mode $x \mapsto \exp(i\lambda_j |x_1|)\phi_j(x_2)$ satisfies the boundedness condition (19). If $\lambda_j^2 < 0$ then $i\lambda_j \in \mathbb{R}_{>0}$ and the corresponding modes satisfies the radiation condition (18). The normal derivative on $\Sigma_\rho^\pm = \{x \in \Omega,\, x_1 = \pm\rho\}$ equals

$$\frac{\partial u}{\partial r}(x) = \pm \frac{\partial u}{\partial x_1}(x) = -\sum_{j=1}^{\infty} \lambda_j \exp(\lambda_j |x_1|) \hat{u}(j) \phi_j(x_2) \qquad \text{for } x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \Sigma_\rho.^\pm.$$

The Dirichlet-to-Neumann operator Λ, defined by

$$\Lambda\psi|_{\Sigma_\rho^\pm} = -\sum_{j=1}^{\infty} \lambda_j \hat{\psi}^\pm(j) \phi_j \qquad \text{for } \psi \in V_2 \text{ such that } \psi|_{\Sigma_\rho^\pm} = \sum_{j=1}^{\infty} \hat{\psi}^\pm(j)\phi_j, \tag{58}$$

is bounded from V_2 into V_2' since $|\lambda_j| \leq (1 + |\lambda_j|^2)$, since

$$\|\Lambda\psi\|_{V_2'}^2 = \sum_{\circledast \in \pm} \sum_{j=1}^{\infty} (1 + |\lambda_j|^2)^{-1/2} \left| \widehat{\Lambda\psi}^{\circledast}(j) \right|^2 \leq \sum_{\circledast \in \pm} \sum_{j=1}^{\infty} (1 + |\lambda_j|^2)^{1/2} |\hat{\psi}^{\circledast}(j)|^2 = \|u\|_{V_2}^2. \tag{59}$$

(In the two-dimensional case this holds even in case that some eigenvalue λ_j^2 vanishes.)

Lemma 6.1. *For $\psi \in V_2$, the function*

$$u(x) = \begin{cases} \sum_{j=1}^{\infty} \hat{\psi}^+(j) \frac{\exp(-\lambda_j x_1)}{\exp(\lambda_j \rho)} \phi_j(x_2) & x_1 > \rho, \\ \sum_{j=1}^{\infty} \hat{\psi}^-(j) \frac{\exp(\lambda_j x_1)}{\exp(-\lambda_j \rho)} \phi_j(x_2) & x_2 < -\rho, \end{cases} \qquad x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \Omega \setminus \Omega_\rho, \tag{60}$$

is the unique weak solution in $H^1_{W,\mathrm{loc}}(\Omega \setminus \Omega_\rho)$ to the Helmholtz equation $\Delta u + (\omega^2/c^2)u = 0$ in $\Omega \setminus \Omega_\rho$ that satisfies the boundary condition $u = \psi$ on Σ_ρ and the radiation and boundedness conditions 24.

Proof. The function u belongs to $H^1_{W,\mathrm{loc}}(\Omega \setminus \Omega_\rho)$ since

$$\begin{aligned} \|u\|^2_{H^1_{W,\mathrm{loc}}(\Omega\setminus\Omega_\rho)} &\simeq \sum_{j\in\mathbb{N}} \int_{\rho<|x_1|<R} (1+|\lambda_j|^2)|\hat{u}(j,x_1)|^2 \, dx_1 \\ &\leq \sum_{j\in\mathbb{N}} (1+|\lambda_j|^2) \left| \int_{\rho<|x_1|<R} \exp(-\lambda_j(|x_1|-\rho)) \, dx_1 \right| |\hat{\psi}(j)|^2 \\ &= 2\sum_{j\in\mathbb{N}} (1+|\lambda_j|^2) \left| \frac{\exp(-2\lambda_j\rho) - \exp(-\lambda_j(R+\rho))}{\lambda_j} \right| |\hat{\psi}(j)|^2 \\ &\leq C \sum_{j\in\mathbb{N}} (1+|\lambda_j|^2)^{1/2} |\hat{\psi}(j)|^2. \end{aligned}$$

Moreover, as in each $j \in \mathbb{N}$, the function $x_1 \mapsto \int_0^H u(x_1,x_2)\phi_j(x_2)\, dx_2$

!!! continue !!!

□

Lemma 6.2. *There are constants $C > 0$ and $c > 0$ such that for $0 < \omega \leq C$ it holds that $-\langle \psi, \Lambda\psi \rangle \geq C\omega \|\psi\|^2_{V_2}$ for all $\psi \in V_2$.*

Proof. As in the proof of Lemma 5.3 we choose $C > 0$ so small that for $0 < \omega \leq C$ no propagating modes exist, i.e., $\lambda_j(\omega) \geq c_1 > 0$ for $0 < \omega \leq C$. Orthogonality of the eigenvectors ϕ_j implies that

$$-\langle \psi, \Lambda\psi \rangle = \sum_{\circledast\in\pm} \sum_{j\in\mathbb{N}} |\lambda_j| |\hat{\psi}^{\circledast}(j)|^2 \sum_{\circledast\in\pm} \geq (1+1/c_1)^{-1} \sum_{j=1}^{\infty} (1+|\lambda_j|^2)^{1/2} |\hat{u}^{\circledast}(j)|^2 = c\|\psi\|^2_{V_2}.$$

□

Lemma 6.3. *Assume that $\omega > 0$ is so large that $J = J(\omega, c, H)$ propagating waveguide modes exist. Then there exists $C = C(\omega) > 0$ such that*

$$-\operatorname{Re}\langle \Lambda\psi\, \psi \rangle \geq -C\|\psi\|^2_{L^2(\Sigma_\rho)} \qquad \text{for all } \psi \in V_2.$$

The proof is analogous to the proof of the corresponding result in three dimensions, i.e., $m = 3$, see Lemma 5.4. As for $m = 3$, the Dirichlet-to-Neumann operator $\Lambda = \Lambda_\omega$ in two dimensions depends analytically on the frequency ω. Proving this result is easier than the corresponding one in Lemma 5.5 since, in two dimensions, Λ does not involve special functions possessing singularities.

Lemma 6.4. *For all $\omega_* > 0$ such that $\lambda_j(\omega_*) \neq 0$ for $j \in \mathbb{N}$ and all $\omega^* > 0$ small enough to satisfy the assumption of Lemma 6.2 there exists an open connected set $U \subset \mathbb{C}$ containing ω_* and ω^* such that $\omega \mapsto \Lambda_\omega$ is an analytic operator-valued function in U.*

7 Existence Theory for Weak Solutions

We have now prepared all tools to provide existence theory for weak solutions of the waveguide scattering problem (22–24).

Assume that $u^i \in H^1_{W,\mathrm{loc}}(\Omega)$ is an incident field that solves the Helmholtz equation (20) and the waveguide boundary conditions. Assume further that $u \in H^1_{W,\mathrm{loc}}(\Omega)$ solves (22) for all $v \in H^1_W(\Omega)$ with compact support and satisfies the radiation conditions (24). Since (22) implies that $\Delta u = \operatorname{div} \nabla u$ is locally square integrable, we acn integrate by parts,

$$0 = \int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)}(1+q)u\overline{v} \right) dx = \int_\Omega \left(\Delta u + \frac{\omega^2}{c^2(x_m)}(1+q)u \right) \overline{v}\, dx$$

for all $v \in H^1(\Omega)$ with compact support in Ω, to show that u satisfies $\Delta + (\omega^2/c^2)(1+q)u = 0$ in the $L^2(\Omega_r)$ for every $r > 0$. Since $\omega^2/c^2(1+q)$ is bounded, we infer that $\Delta u \in L^2_{\mathrm{loc}}(\Omega)$ and elliptic regularity results hence imply that $u \in H^2_{\mathrm{loc}}(\Omega)$. Moreover, $u^s = u - u^i$ is by assumption a radiating function outside of Ω_ρ. Hence, Lemma (5.2) in dimension three, i.e., $m = 3$, and Lemma (6.1) for $m = 2$ imply that

$$\left.\frac{\partial u^s}{\partial \nu}\right|_{\Sigma_\rho} = \Lambda(u^s|_{\Sigma_\rho}) \qquad \text{in } V_m.$$

In consequence, the normal derivative of $u = u^i + u^s$ on Σ_ρ equals

$$\frac{\partial u}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \Lambda\left(u - u^i|_{\Sigma_\rho}\right) \qquad \text{in } V_m.$$

Thus, we multiply the Helmholtz equation by a test function $v \in H^1_W(\Omega_\rho)$ and integrate by parts in Ω_ρ to find that

$$\begin{aligned} 0 &= \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)}(1+q)u\overline{v} \right) dx - \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu} \bar{v}\, ds + \int_{\Gamma_{0,\rho}} \frac{\partial u}{\partial \nu} \bar{v}\, ds + \int_{\Gamma_{H,\rho}} \frac{\partial u}{\partial \nu} \bar{v}\, ds \\ &= \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)}(1+q)u\overline{v} \right) dx - \int_{\Sigma_\rho} \Lambda(u)\, \bar{v}\, ds - \int_{\Sigma_\rho} \left(\frac{\partial u^i}{\partial \nu} - \Lambda(u^i) \right) \bar{v}\, ds, \end{aligned}$$

where ν denotes the unit normal vector corresponding to the boundary of Ω_ρ. Hence, the variational formulation of the waveguide scattering problem (22–24) is to find $u \in H^1_W(\Omega_\rho)$ solving

$$B_\omega(u,v) = \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} (1+q) u \overline{v} \right) dx - \int_{\Sigma_\rho} \Lambda(u)\, \overline{v}\, ds \overset{!}{=} F(v) \tag{61}$$

for all $v \in H^1_W(\Omega_\rho)$, where the continuous anti-linear form F is defined as

$$F(v) = \int_{\Sigma_\rho} \left(\frac{\partial u^i}{\partial \nu} - \Lambda(u^i) \right) \overline{v}\, ds, \qquad v \in H^1_W(\Omega_\rho). \tag{62}$$

Of course, the variational problem (61) can also be considered for arbitrary continuous anti-linear forms $F :\, H^1_W(\Omega_\rho) \to \mathbb{C}$.

Theorem 7.1 (Existence and uniqueness of solution)**.** *(1) The sesquilinear form B_ω in* (61) *and the anti-linear form F in* (62) *are bounded on $H^1_W(\Omega_\rho)$ and B_ω satisfies a Gårding inequality. Thus, the Fredholm alternative holds: Whenever the variational problem* (61) *for $u^i = 0$ possesses only the trivial solution, existence and uniqueness of solution holds for any continuous anti-linear form $F :\, H^1_W(\Omega_\rho) \to \mathbb{C}$.*

(2) There exists $\omega_0 > 0$ such that the variational problem (61) *is uniquely solvable for all incident fields u^i for all frequencies $\omega \in (0, \omega_0)$.*

(3) The variational problem (61) *is uniquely solvable for all incident fields u^i and all frequencies $\omega > 0$ except possibly for a discrete set of exceptional frequencies $\{\omega_j\}_{j=1}^{\mathcal{J}} \subset \mathbb{R}_{>\omega_0}$. If there are infinitely many exceptional frequencies, then $\omega_j \to \infty$ as $j \to \infty$.*

Proof. (1) Due to the boundedness of Λ, see Lemma 5.2 for $m = 3$ and (59) for $m = 2$, and the trace estimate shown in Theorem 4.4, the boundedness of B_ρ and F on $H^1_W(\Omega_\rho)$ follows from

$$|B_\omega(u,v)| \leq \left(1 + \frac{\omega^2}{c_-^2} + \|\Lambda\|_{V_m \to V_m'} \right) \|u\|_{H^1(\Omega_\rho)} \|v\|_{H^1(\Omega_\rho)}, \quad u,v \in H^1_W(\Omega_\rho).$$

Together with the trace estimate $\|\partial u^i / \partial \nu\|_{H^{-1/2}(\Sigma_\rho)} \leq \|\mathrm{div} \nabla u^i\|_{L^2(\Omega_\rho)} \leq (\omega^2/c_-^2) \|u^i\|_{L^2(\Omega_\rho)}$ the same arguments shows that

$$|F(v)| \leq \left[\frac{\omega^2}{c_-^2} \|u^i\|_{L^2(\Omega_\rho)} + \|\Lambda\|_{V_m \to V_m'} \|u^i\|_{H^1(\Omega_\rho)} \right] \|v\|_{H^1(\Omega_\rho)}, \quad v \in H^1_W(\Omega_\rho).$$

The Gårding inequality for small frequencies follows from the lower bound of Λ at arbitrary frequencies. First, the assumption that $c_- < c(x_m)$ for $x_m \in (0,H)$ yields

$$\mathrm{Re}(B_\omega(u,u)) \geq \|u\|^2_{H^1_W(\Omega_\rho)} - \left(\frac{\omega^2}{c_-^2} (1 + \|q\|_{L^\infty(\Omega_\rho)}) + 1 \right) \|u\|^2_{L^2(\Omega_\rho)} - \mathrm{Re} \left(\int_{\Sigma_\rho} \Lambda u\, \overline{v}\, ds \right)$$

for arbitrary $u \in H^1_W(\Omega_\rho)$. Second, Lemma 5.4 for $m = 3$ and Lemma 6.3 for $m = 2$ imply that $-\operatorname{Re}(\int_{\Sigma_\rho} \Lambda(u)\,\bar{v}\,ds) \geq -C\|u\|^2_{L^2(\Sigma_\rho)}$ for some constant $C > 0$. Thus,

$$\operatorname{Re}(B_\omega(u,u)) \geq \|u\|^2_{H^1_W(\Omega_\rho)} - \left(\frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(\Omega_\rho)}) + 1\right)\|u\|^2_{L^2(\Omega_\rho)} - C\|u\|^2_{L^2(\Sigma_\rho)}.$$

Since the embedding of $H^1_W(\Omega_\rho)$ in $L^2(\Omega_\rho)$ is compact and since further the trace operator from $H^1_W(\Omega_\rho)$ into $L^2(\Sigma_\rho)$ is compact due to the compact embedding of $H^{1/2}(\Sigma_\rho)$ in $L^2(\Sigma_\rho)$ the latter estimate is indeed a Gårding inequality for the form B_ω. In consequence, the variational problem (61) is Fredholm of index zero. In particular, uniqueness of solution implies existence of solution together with the continuous dependence of the solution on the right-hand side F.

(2) The L^2-coercivity of Λ shown in Lemma 6.2 and Lemma 5.3 for $m = 3$ and $m = 3$, respectively, implies that Λ is a positive operator for small frequencies. Thus,

$$\operatorname{Re}(B_\omega(u,u)) \geq \|\nabla u\|^2_{L^2(\Omega_\rho)} - \frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(\Omega_\rho)})\|u\|^2_{L^2(\Omega_\rho)} + c\omega\|u\|^2_{L^2(\Sigma_\rho)}, \quad u \in H^1_W(\Omega_\rho).$$

Poincaré's inequality states that $\|u\|^2_{L^2(\Omega_\rho)} \leq (H^2/2)\|\nabla u\|^2_{L^2(\Omega_\rho)^m}$ for all $u \in H^1_W(\Omega_\rho)$. In consequence,

$$\operatorname{Re}(B_\omega(u,u)) \geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega_\rho)} + \frac{1}{H^2}\|u\|^2_{L^2(\Omega_\rho)} - \frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(\Omega_\rho)})\|u\|^2_{L^2(\Omega_\rho)}$$

and the left-hand side is equivalent to $\|u\|^2_{H^1_W(\Omega_\rho)}$ if $\omega^2 < c_-^2(1 + \|q\|_{L^\infty(\Omega_\rho)})/H^2$. Thus, if ω is small enough to satisfy this bound then B_ω is coercive on $H^1_W(\Omega_\rho)$ and the lemma of Lax and Milgram implies that (61) is uniquely solvable for any right-hand side.

(3) Part (2) of the proof shows that (61) is uniquely solvable whenever $\omega > 0$ is less than some $\omega_0 > 0$. If ω is larger than or equal to ω_0 we exploit that the operator-valued function $\omega \mapsto \Lambda_\omega$ depends analytically on ω. More precisely, fix an arbitrary $\omega^* > 0$ such that $\lambda_j^2(\omega) = 0$ and some $\omega_* \in (0, \omega_0)$. Depending on the dimension $m = 2, 3$, either Lemma 5.5 or Lemma 6.4 show that there exists an open connected set $U \subset \mathbb{C}$ containing ω_* and ω^* such that $\omega \mapsto \Lambda_\omega$ is an analytic operator-valued function in U. In consequence, the entire sesquilinear form

$$B_\omega(u,v) = \int_{\Omega_\rho}\left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)}(1+q)u\overline{v}\right)dx - \int_{\Sigma_\rho} \Lambda(u)\,\bar{v}\,ds$$

depends analytically on ω in U. Moreover, choosing the frequency $\omega_* \in U$, the variational problem (61) is uniquely solvable due to part (2) of this theorem. Hence, analytic Fredholm theory implies that problem (61) is uniquely solvable for all $\omega \in U$ except possibly for a countable sequence of exceptional frequencies without accumulation point in U. In particular, there exists at most a countable set of real frequencies where uniqueness of solution fails. If there exists an infinite set of real-valued exceptional frequencies then these frequencies necessarily tend to infinity. $\square$

Remark 7.2. *Analytic Fredholm theory is not able to prove uniqueness of solution for those frequencies where some eigenvalue $\lambda_j^2(\omega)$ vanishes; this does however not imply that uniqueness of solution does indeed fail at those frequencies, compare [AGL08].*

Theorem 7.3. *Assume that Assumption 3.1 holds. (1) If the variational problem* (61) *is uniquely solvable for any incident fields u^i, then any solution $u \in H^1_W(\Omega_\rho)$ can be extended to a weak solution $\tilde{u} \in H^1_{\mathrm{loc}}(\Omega)$ of the waveguide scattering problem (22–24) by setting $\tilde{u}|_{\Omega_\rho} = u|_{\Omega_\rho}$ and*

$$\tilde{u}(x) = u^i(x) + \sum_{j=1}^{\infty} \sum_{n \in \mathbb{Z}} \hat{u}(j,n) \frac{H_n^{(1)}(i\lambda_j r)}{H_n^{(1)}(i\lambda_j \rho)} \exp(in\varphi) \phi_j(x_3) \quad \text{for } x = \left(\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3 \end{smallmatrix}\right) \text{ in } \Omega \setminus \overline{\Omega_\rho}, \tag{63}$$

where the coefficients $\hat{u}(j,n)$ are defined by

$$\hat{u}(j,n) = \int_0^H \int_0^{2\pi} (u - u^i) \left(\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3 \end{smallmatrix}\right) e^{-in\varphi} \phi_j \, d\varphi \, dx_3, \qquad j \in \mathbb{N},\, n \in \mathbb{Z}. \tag{64}$$

Moreover, $\tilde{u}$ is the unique weak solution to the waveguide scattering problem (22–24).

(2) If $\mathrm{Im}(q) \geq c_0 > 0$ on a non-empty open subset D of Ω_ρ, then there are no exceptional frequencies, i.e., the variational problem (61) *and the scattering problem (22–24) are both uniquely solvable for all incident fields u^i and all frequencies $\omega > 0$.*

Proof. (1) Assume that $u \in H^1_W(\Omega_\rho)$ is the unique solution (61). As in the beginning of this section, we note that (61) implies that $\operatorname{div} \nabla u \in L^2(\Omega_\rho)$. Choosing $v \in H^1_W(\Omega_\rho)$ such that $v|_{\Sigma_\rho} = 0$ we can hence integrate by parts in (61), to find that

$$\begin{aligned} 0 &= \int_{\Omega_\rho} \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} (1+q) u \overline{v} \right) dx \\ &= - \int_{\Omega_\rho} \left(\Delta u + \frac{\omega^2}{c^2(x_m)} (1+q) u \right) \overline{v} \, dx + \int_{\Gamma_H \cap \{|\tilde{x}| < \rho\}} \frac{\partial u}{\partial x_m} \overline{v} \, ds. \end{aligned}$$

Integrating now a second time by parts for a test function $v \in H^1_W(\Omega_\rho)$ and exploiting the definition of the right-hand side $F(v)$ in (62) then shows that

$$\int_{\Sigma_\rho} \left(\frac{\partial u}{\partial \nu} - \Lambda(u) \right) \overline{v} \, ds = \int_{\Sigma_\rho} \left(\frac{\partial u^i}{\partial \nu} - \Lambda(u^i) \right) \overline{v} \, ds \tag{65}$$

for all $v \in H^1_W(\Omega_\rho)$. We define $u^s \in H^1_W(\Omega_\rho)$ by $u = u^i + u^s$ and note that (65) implies that the equation $(\partial u^s/\partial \nu)|_{\Sigma_\rho} = \Lambda(u^s|_{\Sigma_\rho})$ holds in V_m.

Let us in the following indicate by $(\cdot)|^{\pm}_{\Sigma_\rho}$ if a trace on Σ_ρ is taken from the inside (-) or from the outside (+) of Ω. Further we define u^s in $\Omega \setminus \overline{\Omega}_\rho$ by the series in (63) such that $\tilde{u} = u^i + u^s$ holds in $\Omega \setminus \overline{\Omega}_\rho$.

By the trace trace estimate from Theorem 4.4 and the representation of functions in V_m we note that the coefficients $\hat{u}(j,n)$ in (64) are defined such that

$$(u - u^i)\big|^{-}_{\Sigma_\rho}(x) = \sum_{j=1}^{\infty} \sum_{n \in \mathbb{Z}} \hat{u}(j,n) \exp(in\varphi) \phi_j(x_3), \quad x = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ x_3 \end{pmatrix} \in \Sigma_\rho,$$

holds in V_3. This implies that that $u|^{-}_{\Sigma_\rho}$ equals the restriction $\tilde{u}|^{+}_{\Sigma_\rho}$, i.e., the extension $\tilde{u}$ is continuous over Σ_ρ in the trace sense. By construction of Λ and $\tilde{u}$, it follows from Lemma (5.2) in dimension three and Lemma (6.1) in dimension two that $\tilde{u}$ is a radiating solution to the Helmholtz equation in $\Omega \setminus \overline{\Omega}_\rho$ with normal derivative

$$\frac{\partial \tilde{u}}{\partial \nu}\bigg|^{+}_{\Sigma_\rho} = \left[\frac{\partial u^i}{\partial \nu} + \frac{\partial u^s}{\partial \nu}\right]\bigg|^{+}_{\Sigma_\rho} = \left[\frac{\partial u^i}{\partial \nu}\right]\bigg|^{+}_{\Sigma_\rho} + \Lambda(u^s|^{+}_{\Sigma_\rho}) = \left[\frac{\partial u^i}{\partial \nu}\right]\bigg|^{-}_{\Sigma_\rho} + \Lambda(u^s|^{-}_{\Sigma_\rho})$$

since we already showed above that $(\partial u^s/\partial \nu)|^{-}_{\Sigma_\rho} = \Lambda(u^s|^{-}_{\Sigma_\rho})$. Since the normal derivative of $\tilde{u}$ across Σ_ρ is hence also continuous in the trace sense, the latter function is a weak solution in $H^1(\Omega)$ to the Helmholtz equation in all of Ω. In consequence, $\tilde{u}$ solves the waveguide scattering problem (22–24). Note that interior elliptic regularity results [McL00, Chapter 4] show that $\tilde{u} \in H^2_{\mathrm{loc}}(\Omega)$.

Finally, uniqueness of this scattering problem finally follows from uniqueness of solution of the variational problem (61), since any non-trivial solution to the scattering problem for $u^i = 0$ is a non-trivial solution to the variational problem with vanishing right-hand side.

(2) Assume that $\mathrm{Im}(q) > 0$ on a non-empty open subset $D \subset \Omega_\rho$ and consider a solution $u \in H^1_W(\Omega_\rho)$ to (61) with vanishing right-hand side $F = 0$ or, equivalently, vanishing incident field $u^i = 0$. Since u^i vanishes, the first part of this proof shows that $\partial u/\partial \nu = \Lambda(u)$ on Σ_ρ. We extend the solution u to all of Ω using the formula (63) and, by abuse of notation, call the extended function again u. Recall from part (1) that this extension belongs to $H^2_{\mathrm{loc}}(\Omega)$ and is a radiating solution to the Helmholtz equation $\Delta u + \omega^2/c^2(x_m)(1+q)u = 0$ in Ω. Since ν is the exterior unit normal to Ω_ρ, taking the imaginary part of (61) with $v = u$ shows

that

$$
\begin{aligned}
0 = \operatorname{Im} B_\omega(u,u) &= -\int_{\Omega_\rho} \frac{\omega^2 \operatorname{Im}(q)}{c^2(x_m)} |u|^2 \, dx - \operatorname{Im} \int_{\Sigma_\rho} \Lambda(u)\, \bar{u} \, ds \\
&= \int_{\Omega_\rho} \frac{\omega^2 \operatorname{Im}(q)}{c^2(x_m)} |u|^2 \, dx - \operatorname{Im} \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu} \, \bar{u} \, ds \\
&= \int_{\Omega_\rho} \frac{\omega^2 \operatorname{Im}(q)}{c^2(x_m)} |u|^2 \, dx + \operatorname{Im} \int_{\Omega_r \setminus \overline{\Omega_\rho}} \left[|\nabla u|^2 - \frac{\omega^2}{c^2(x_m)} |u|^2 \right] \, dx - \operatorname{Im} \int_{\Sigma_r} \frac{\partial u}{\partial \nu} \, \bar{u} \, ds.
\end{aligned}
$$

The second-to-last term of the last equation obviously vanishes. We investigate the last term, relying on the orthonormal expansion $u(x) = \sum_{j=1}^\infty \hat{u}(j,\tilde{x})\phi_j(x_m)$ valid in Ω,

$$
\int_{\Sigma_r} \frac{\partial u}{\partial \nu} \, \bar{u} \, ds = \sum_{j=1}^\infty \int_{|\tilde{x}|=r} \frac{\partial \hat{u}(j,\tilde{x})}{\partial \nu} \hat{u}(j,\tilde{x}) \, ds.
$$

Since u is a radiating solution to the Helmholtz equation, the two-dimensional function $\tilde{x} \mapsto \hat{u}(j,\tilde{x})$ solves the Helmholtz equation $(\Delta_{\tilde{x}} - \lambda_j^2)\hat{u}(j,\tilde{x}) = 0$ for $|\tilde{x}| > \rho$. Moreover, for $1 \leq j \leq J$, the wave number $i\lambda_j > 0$ of the latter Helmholtz equation is positive and $\hat{u}(j,\cdot)$ satisfies Sommerfeld's radiation condition; for $j > J$ the wave number $i\lambda_j \in i\mathbb{R}_{>0}$ of the latter Helmholtz equation is purely imaginary and $\tilde{u}(j,\cdot)$ is bounded in $|\tilde{x}| > \rho$. For the solutions $\hat{u}(j,\cdot)$, $1 \leq j \leq J$, to the Helmholtz equation with positive wave number that satisfy Sommerfeld's radiation condition it is well-known that

$$
\operatorname{Im} \int_{|\tilde{x}|=r} \frac{\partial \hat{u}(j,\tilde{x})}{\partial \nu} \hat{u}(j,\tilde{x}) \, ds \geq 0, \qquad 1 \leq j \leq J,
$$

since the latter expression equals (a constant times) the L^2-norm of the far field pattern of $\hat{u}(j,\cdot)$ (see, e.g.,[CK12]). For $j > J$, $-\lambda_j^2$ is negative and the solution $\hat{u}(k,\cdot)$ is a bounded solution to the latter equation; since $i\lambda_j \in i\mathbb{R}_{>0}$ is the wave number of the corresponding Helmholtz equation, this bounded solution, together with all its derivatives decays exponentially. (If $m = 3$, this also follows from the estimates of the Hankel functions in the proof of Lemma 5.2; if $m = 2$, then the exponential decay is obvious from the series representation of the solution, compare (59).) Thus,

$$
\operatorname{Im} \int_{|\tilde{x}|=r} \frac{\partial \hat{u}(j,\tilde{x})}{\partial \nu} \hat{u}(j,\tilde{x}) \, ds \to 0 \qquad \text{as } r \to \infty, \qquad j > J.
$$

Choosing $r > 0$ large enough, we hence conclude by our assumption on $\operatorname{Im}(q)$ that

$$
0 = \operatorname{Im} B_\omega(u,u) = \int_{\Omega_\rho} \frac{\omega^2 \operatorname{Im}(q)}{c^2(x_m)} |u|^2 \, dx + \operatorname{Im} \int_{\Sigma_r} \frac{\partial u}{\partial \nu} \, \bar{u} \, ds \geq c_0 \int_D \frac{\omega^2}{c^2(x_m)} |u|^2 \, dx \geq 0.
$$

Thus, u vanishes on the open, nonempty set D. Now, the unique continuation property for solutions of the Schrödinger-type equation $\Delta u + (\omega^2/c^2(x_m))(1+q)u = 0$, see [JK85, Theorem 6.3, Remark 6.7], implies that u vanishes in all of Ω. Thus, uniqueness of solution to (61) holds and implies by Theorem 7.1 that (61) is uniquely solvable for all right-hand sides. Part (1) of this theorem then yields the claim. □

Remark 7.4. *If the inhomogeneous medium described by the contrast q is replaced by an impenetrable obstacle $D \subset \Omega_\rho$ with either Dirichlet, Neumann or impedance boundary condition, then the variational problem for the total field is posed in $V = H^1_W(\Omega_\rho \setminus \overline{D})$ for a Neumann or impedance boundary condition. For a Dirichlet boundary condition, functions in V additionally have to satisfy a Dirichlet boundary condition on ∂D. The results of Theorem 7.1 and Theorem 7.3(1) hold analogously.*

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